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Gert Schubring

Analysing Historical Mathematics Textbooks

Springer

International Studies in the History of Mathematics and its Teaching

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Portrait of Sylvestre-François Lacroix

Preface

This book, studying the history of a specific kind of mathematical book, has itself a history.

The origin of this book was a lecture course on the history of mathematical textbooks, given in July and August of 1995 in the Mathematics Department at *Pontifícia Universidade Católica do Rio de Janeiro* (PUC Rio). The lecture course was a part of my stay as a visiting professor at PUC Rio, following an invitation by João Bosco Pitombeira de Carvalho.

Later on, I organised this course as Lecture Notes in English, as I had taught them. They were published in the series of pre-prints of the Mathematics Department at PUC Rio, in 1997. A second edition came out, also there, in 1999, as the outcome of a workshop which I held at the *III Seminário Nacional de História da Matemática*, in Vitória, ES (Brazil), March 1999. That new edition prompted me to enlarge in particular the analysis of Lacroix's textbook production. Due to my increasing activities as a visiting professor at various Brazilian universities and the need of Brazilian students to have texts available in Portuguese, I transformed the pre-print versions into a proper book in that language, enlarged and enriched by numerous illustrations. The book, entitled *Análise Histórica de Livros de Matemática: Notas de Aula*, was published by *Editora Autores Associados*, in Campinas, SP (Brazil), 2003.

Since then, however, colleagues in non-Lusophone countries regretted not having this book available in English for their lecture courses. Eventually, and thanks to this series by Springer, I re-transformed the Portuguese book into an again-revised, enriched, and updated English book.

I should comment upon the textbook as a specific kind of mathematical publication, distinguishing it from any other in the genre. A first distinction is already evoked by the citation at the beginning of Chap. 1 where Roger Martin emphasised the difference between a compendium or handbook and a textbook for teaching and learning some topics in mathematics. A highly emblematic example for this difference is the twofold form Lacroix used to publish his *Differential and Integral Calculus*: first as a 1832-page-long *traité*, in three volumes, published between 1797 and 1800, comprising all knowledge he thought to be significant for this subject. Shortly afterwards, in 1802, Lacroix transformed his *traité* into a 574-page *Traité élémentaire* for his lectures at *École Polytechnique*.

Another distinction was Thomas Kuhn's characterisation of publications, namely between research publications, mainly in journal papers, and textbooks, characterised by Kuhn as "normalising" science (see Chap. 1).

Given these two distinctions, the genre "textbook" will prove in this book to be a highly complex and multifaceted subject. It arose as a "one-volume" manuscript since Antiquity, preserved in the most varied forms of material: from well-conservable material like cuneiform tablets to perishable material like bamboo slats, papyrus rolls, or palm leaves. Thereafter, this unique form of text was printed on paper. Originally destined for just one institutionalised level of teaching, the evolution of teaching systems led to vastly differentiated forms of textbooks for the likewise differentiated levels of teaching. And as the textbook triangle will evoke, its variety is also intended to meet different addressees – mainly teachers and students. While the genre remained rather stable for extended periods, textbooks addressed to students were again and strongly differentiated due to technological innovations – it transgresses the so far traditional printed forms now including various digital devices.

Historiography of mathematics has always brought attention to textbooks, even when they were not the main focus of a research. For mathematics education, contrariwise, schoolbooks are a key issue, concerning development of teaching material and research, as well as the history of these books. A first initiative for establishing international studies had been undertaken by the late John Fauvel (1947–2001). For a textbook colloquium he was preparing together with the *British Society for the History of Mathematics*, he elaborated the booklet *Mathematics Textbooks*. *An Annotated Bibliography of Historical Studies*, printed internally in 2002. In its 15 pages, the booklet lists 70 publications – predominantly on some aspects of the history of mathematics with a certain attention to textbooks, and British authors. In 2005, the BSHM carried out a *textbook colloquium – Mathematical Textbooks: History, Production and Influence* – as a first international attempt to study the history of mathematics textbooks (see report by Richard Simpson, 2006). The ICMT conference series – International Conference on Mathematics Textbook Research and Development – features historical perspectives on textbooks as one of its thematic issues, since ICMT 2, in 2017.

This book is an attempt to analyse the complex development of textbook production and reception since Antiquity, attempting in particular to take account of differing forms of the textbook triangle in various cultures since then.

Rio de Janeiro, Brazil Gert Schubring
June 2022

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Chapter 1
Why Study Historical Books Intended for Teaching?

It should not be forgotten, [...] that an elementary book on any science differs essentially from a complete treatise on the same subject; that each one has its own particular objective, its march and its different results. The one, in order to offer a complete body of the doctrine, must dedicate itself to all the details, exhaust the consequences, and not ignore any known truth about the art or science it exposes. The other, on the contrary, only destined to lay the foundations rather than to erect the building, comprises the principles and makes of them a strict choice; it devotes little to applications, and develops only the important points, the first truths upon which the whole science depends.

Roger Martin.

Élémens de Mathématiques. Paris, an X (1802). (Roger Martin (1741–1811), mathematics teacher at the Collège Royal in Toulouse, was the author of a textbook, first published in 1781 and with a second edition in 1802, remarkable for its contributions to algebraisation (see Schubring 2005, pp. 116 ff.))

Traditionally, books intended for teaching were not a common topic or standard in the history of science. They were treated more or less with disdain, being considered uninteresting or even boring. Libraries used not to collect and store books for teaching, in particular schoolbooks. Historiographical interest tended to focus on the achievements of the most notable scientists. Sociological reasons may be responsible for such preferences: part of the fame used to pass on to the historian who studied a very important scholar, and there were no comparable rewards for those studying textbooks and their authors.

However, after Thomas Kuhn published his book "The Structure of Scientific Revolutions" in 1962, the situation changed profoundly. One of Kuhn's innovations was the introduction of an important differentiation. Rather than seeing the

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evolution of science as a cumulative process that progresses through both continuous effort and consecutive leaps, he was the first to distinguish between periods of normal, "banal" science and revolutionary periods. According to Kuhn, a normal period is characterised by a stable paradigm, in which scientists are concerned with "solving problems" within the framework of this paradigm. In contrast, a revolutionary period consists of first challenging and then changing the dominant paradigm.

It can be said that Thomas Kuhn's focus and its consequences have improved the status of normal science in historiography. Part of the attention hitherto almost exclusively dedicated to the outstanding scientist of the scientific discipline in question has shifted to the average researcher. As a prototype of a worker within the limits of a paradigm, the average (or "normal") researcher came to be discovered as a valuable and important topic of historical research as well. At the same time, Kuhn introduced a social perspective into historical analysis with his concept of a "scientific community", the name he gave to the academic group that decides whether new scientific theories or new results will be accepted and recognised. His introduction of this concept can be seen as a negation of the old "objectivity", and as an attempt to link the development of science to that of social and cultural values. It can be a challenge to find out if there have been revolutions in mathematics as well. The best of the most recent presentations in this debate is the one edited by Donald Gillies (1992).

Textbooks entered the scene as legitimate objects of historical research after Kuhn discussed them at length, although belittling them:

> The more rigid definition of the scientific group has other consequences. When the individual scientist can take a paradigm for granted, he needs no longer, in his major works, attempt to build a field anew, starting from the first principles and justifying the use of each concept introduced. That can be left to the writer of textbooks. Given a textbook, however, the creative scientist can begin his research where it leaves off and thus concentrate exclusively upon the subtlest and most esoteric aspects of the natural phenomena that concern his group (Kuhn 1962, pp. 19–20).

and:

> Today in the sciences, books are usually either texts or retrospective reflections upon one aspect or another of the scientific life. The scientist who writes one is more likely to find his professional reputation impaired than enhanced. Only in the earlier, pre-paradigm, stages of the development of the various sciences did the book ordinarily possess the same relation to professional achievement that it still retains in other creative fields (ibid., p. 20).

Kuhn sees textbooks as introductions to the normal science paradigm in question, presenting the principles and elements – their foundations and their main body of knowledge. He attributes to textbooks the separation between "school knowledge" and "scientific knowledge", admitting that school knowledge is essential for research and arises from it.

Kuhn makes a clear distinction between the types of text in textbooks and research papers; this difference leads him to segregate the lay public and professionals. In the chapter on "the invisibility of revolutions", he essentially attributes to textbooks the function of "normalising" and conveying the impression that the character of science is cumulative and non-revolutionary:

> For reasons that are both obvious and highly functional, science textbooks (and too many of the older histories of science) refer only to that part of the work of past scientists that can easily be viewed as contributions to the statement and solution of the texts' paradigm problems. Partly by selection and partly by distortion, the scientists of earlier ages are implicitly represented as having worked upon the same set of fixed problems and in accordance with the same set of fixed canons that the most recent revolution in scientific theory and method has made seem scientific. No wonder that textbooks and the historical tradition they imply have to be rewritten after each scientific revolution. And no wonder that, as they are rewritten, science once again comes to seem largely cumulative (ibid., p. 138).

Textbooks making the development of science linear are, hence, hiding revolutions which had occurred:

> The result is a persistent tendency to make the history of science look linear or cumulative, a tendency that even affects scientists looking back at their own research (ibid., p. 139).

Even considering that it is certainly an exaggeration to ascribe so many additional functions to textbooks, let alone so many negative ones, the question about the effective existence of such functions certainly makes it worthwhile to examine textbooks and their production modes in more specific detail. Our proper issue will be to discuss whether Kuhn's propositions remain valid for mathematics. In reality, there are mathematics textbooks that are profoundly revolutionary, like those by van der Waerden, Bourbaki, etc.

Despite Kuhn's own slur on textbooks, his work paved the way for accepting them as a rewarding study subject in the history of science. An important approach in this study is to find the roots of the *new* within the *old*: this approach helps to examine both the education and training of "normal" and of "revolutionary" scientists. To study the new, textbooks are indispensable. However, their analysis presents enormous methodological problems.

1.1 Textbooks as Interface

Textbooks constitute a decisive element in the interface between research and teaching. Given the more intense interests in history of science for social history and hence for the interface between research and teaching, there have been more discussions and conceptualisations in the last two decades.

One of them was the discussion between Bruno Belhoste and me exactly about the interface between research and teaching. Belhoste had published a strong plea for a reassessment of the role of teaching in the history of mathematics. He criticised historians' abstinence of mathematics in addressing this issue and researching the social and intellectual space in which the production of mathematics occurs, affirming that there exists no completely autonomous sphere of theoretical production (Belhoste 1998, p. 289). For Belhoste, teaching was understood as a special modality of the socialisation of knowledge in which the recipient finds himself in the situation of learning. Of the three major directions for research into these interfaces, the third was the most challenging one: to investigate the impact of teaching upon the development and diffusion of mathematical practices (ibid., p. 289 and passim).

In my answer to Belhoste's paper, I had agreed with this research programme, and in particular with his critique of "the wrong idea that mathematical production can be separated *a priori* by the historian from the conditions of its reproduction" (Schubring 2001, p. 298). Yet, to identify research and teaching with the two poles of production and reproduction would imply a hierarchy between invention and transmission, where production is attributed the primary status and teaching a derived status. This would impede conceptualising contributions of teaching to research. Rather, one might qualify it as a chicken and egg problem, whether there has been first teaching or whether research emerged first.

As a consequence, research and teaching prove to be so intimately interconnected that neither of the sides can be isolated. In fact, approaches from the social history of science can be used to unravel such interactions. Studies of institutionalisation of science and professionalisation of its practitioners show the formation of basic conceptions, of paradigms, within a given scientific community. Jean Piaget and Rolando Garcia had emphasised the indissoluble connection between the epistemology of a scientific discipline and its socio-cultural embedding:

> In our view, at each moment in history and in each society, there exists a dominant epistemic framework, a product of social paradigms, which in turn becomes the source of new epistemic paradigms. Once a given epistemic framework is constituted, it becomes impossible to dissociate the contribution of the social component from the one that is intrinsic to the cognitive system. That is, once it is constituted, the epistemic framework begins to act as an ideology which conditions the further development of science (Piaget and Garcia 1989, p. 255).

I had commented the intimate interface:

> To analyse mathematical production, it is therefore necessary to investigate the functions that mathematics exerts in relation to other subsystems in a given period, culture, and state (Schubring 2001, p. 302). Clearly, the educational system, in particular the systems of secondary schooling and higher education, exerts key roles. Even academies and research institutes are not autonomous systems, but subsystems related to the professionalisation of scientists (Schubring 2019, p. vii).

1.2 Social History

Textbooks constitute a major source for investigating the interface between research and teaching. Undertaking such investigations presents, therefore, methodological challenges which were hitherto, however, seldomly taken into account.

A traditional approach had been to consider it sufficient to analyse an isolated textbook, in a purely internal way, that is, to evaluate its internal structure. A serious historian, however, will not be satisfied with such descriptive data; on the contrary, he or she should aspire to deepen the understanding of this structure and the established internal connections, hence situating the author and his or her work in the context of the development of mathematics. The historian should also be interested in evaluating the originality of the author's contributions to this development. An assessment may, at first sight, not seem to be very difficult for scientific mathematical texts, since the overall development of mathematics is well known and bearing in mind that the impact of any particular work can be analysed, say, by computing the number of references made to it in subsequent publications. However, objections to this practice are not only due to the fact that there are mathematical works that have remained virtually unknown for several generations, but also what this perspective implies: it presupposes a linear and continuous development of mathematics, and possibly does not properly consider minor or forgotten approaches and traditions.

The methodological challenges can be elucidated more markedly if we restrict ourselves for the moment to textbooks intended for use in schools, to schoolbooks. Schoolbooks demonstrate that the outlined manner for the analysis of texts is insufficient, since the standard against which they must be analysed (the "*étalon*") is the body of *school mathematics* and its development. However, the number of authors who produce textbooks is enormous – one can ask whether their number is comparable to that of researchers – and this makes it even more difficult to assess the originality of the contribution of these books compared to that of mathematical research *stricto sensu*. Furthermore, it must be frankly admitted that very little is known about the constitution and development of school mathematics. There is not only the absence of an established standard (*étalon*) against which school mathematics could be assessed, but the task becomes even more complex if we consider the enormous variability of "taught" mathematics – both elementary and higher – a variability caused, in fact, by cultural and social variables dominating in the numerous countries.

To illustrate the extension of this great variability, a few examples will be briefly outlined. In nineteenth century France, a "Mathematics for all", of low standard, coexisted with a "Mathematics for candidates for the *Grandes Écoles*". In Germany, throughout the nineteenth century, school mathematics reveals not only evolution and frequent changes, but also breaks and ruptures. Subsequently, divergent patterns materialised in different types of school mathematics according to the type of school: Gymnasium mathematics, *Realschule* mathematics, vocational school mathematics, and primary school mathematics as a particularly hermetically closed body of knowledge.

Textbooks for higher education likewise do not correspond to a unique body of mathematical knowledge. Higher education also constitutes a highly diversified institutional field: with differing attributions of functions for professional formation and ensuing variable conceptions of knowledge to be taught.

Thus, since there is no direct access to an immediate internal interpretation of a textbook, it is imperative to analyse it as a part of a broader socio-cultural context.

1.3 Methods for Textbook Analysis

1.3.1 What Is Meant by 'Context'?

Particularly important contributions have been provided by the book *Scientific Sources and Teaching Contexts Throughout History*, edited by Alain Bernard and Christine Proust in 2014. Their starting point is the gap between the importance or even necessity attributed in history of science, since some decades, to the investigation of "the cultural and intellectual contexts in which the sources were produced and used", and two basic facts:

> The first is that, too often, sources of scientific knowledge have been studied without taking into account the various 'contexts' of transmission, specifically of the "teaching context" in which this knowledge was elaborated, used and transmitted. The second fact is that other sources have been considered – sometimes dismissively and sometimes mistakenly – as relating to teaching and learning activities with little attempt to offer precision on, and demonstration of, the existence and nature of the underlying 'teaching context' (Bernard and Proust 2014a, b, c, p. 1).[1]

The editors, both historians of mathematics of the Antiquity – Greek and Babylonian mathematics, respectively – started their project from the question: "could historians of ancient science have missed something important by neglecting the teaching context when studying their sources?". Thus, they were led to the dual problem of: "either ignoring the teaching context when it should be taken into account, or taking it for granted when it should be justified and studied more carefully" (ibid., p. 2). The pertinence of this very trifling observation will be encountered at various instances in the present investigation. The authors searched which kinds of sources can provide evidence about teaching and learning practices and were thus led to refine and detail what is meant by 'context' – an issue clearly of utmost importance for concretising what is meant by 'contextualising', beyond the most basic meaning: namely referring "to the 'textual' surroundings of a given word or passage and that contributes to defining or changing its meaning" (ibid., p. 3). They emphasised the following dimensions of meaning which should be covered by 'context'.

- Firstly, there is the meaning of *environment*: 'context' would primarily mean "a general set of cultural and traditional values, received norms and habits that define a general environment and a system of shared objectives for the activities and sources under scrutiny".
- A second meaning is as *situation*: the term can "refer to a set of concrete gestures, procedures and codified practices that are attested in varying degrees by the written sources".
- The third meaning is related to a *meaningful textual system*: "is understood in a very strict sense that is close to its etymological meaning (*contextus*), the word

[1] Karine Chemla has commented upon this crucial difference in her chapter in the same volume: "Note, however, that the fact that these books were used as textbooks does *not* necessarily mean that they had been written *for* educational purposes" (Chemla 2014, p. 309).

> 'context' might be understood as the *whole system of textual sources, as well as intellectual activities* that makes a set of signs, meaningless in itself, a coherent and meaningful entity. This coherence might be attained by the fact that texts or pieces of text are used, composed or read in parallel to other texts that together build meaning and direction, like a traditional text accompanied by its marginalia, or the tablets that had to be used in a concomitant way in the elementary learning of mathematics in the Old Babylonian period" (ibid., pp. 3 f.).

The epistemic framework, pointed out by Piaget and Garcia, belongs to the environment meaning. For textbook analysis, it is in particular the second meaning, as situation, which will prove to be enormously pertinent, by revealing the institutional context, and by referring to several subsystems of a given society: the educational, the science production, and the labour market subsystems and their interactions. The third meaning is highly relevant by showing that none of the textbooks is a "stand-alone" document, but rather they belong to a possibly quite complex system of texts, beginning by a series within which the book in question might be elaborated or to an entire tradition of interrelated texts.

1.3.2 Hermeneutics

1.3.2.1 The Development of Hermeneutics

What is at stake, in this book is, therefore, a contextualised analysis of historical textbooks. How to realise such a task? As a matter of fact, the most general approach for text analysis is *hermeneutics*. This term might still sound unknown or rather related to other disciplines like theology or philology, but it constitutes, in fact, the adapted approach as should be explained here somewhat, for better familiarising this notion. Among those who already have some knowledge of it, there might be a misunderstanding – probably quite widespread – of it being just a method for an internal textual analysis.

Hermeneutics is a methodology with a long scientific tradition for analysing texts to reveal their meaning. The term has its origin in a Greek word: ἑρμηνεύειν, or in Latin letters *hermēneúein*. In the Greek language this word means: to explain, to interpret. It is worth emphasising that hermeneutics owes its origins to Modern Times. Only with the end of the Middle Ages, when knowledge had to basically remain stable and standards and doctrines were prescribed, was it possible to realise a radically different practice: no text did remain sacrosanct – doubting was established as a methodological approach.

Characteristically, sacred texts have since then been investigated with the methods of hermeneutics. No longer considered to be exclusively emanating from divine revelation, it became possible to detect different layers in Bible texts – meaning different authors. In the books of Genesis in Hebrew, for example, different names used for 'God' have been identified in different parts of the text: Jahve, Elohim. The first protagonist who applied such methods to interpret the Old Testament was the Dutch Jew (emigrated from Portugal) Baruch Spinoza (1632–1677). But even so, the practice of such hermeneutics was not yet evident: Baruch was expelled from his Jewish community in Amsterdam.

After this first stage, the application of hermeneutics moved to focus on the classical texts of Antiquity, by Latin authors and in particular by Greek authors: when they arrived in Europe as manuscripts, in particular since the end of the Middle Ages, they had been copied innumerous times by scribes, thus the texts were in large parts corrupted. Philologists, who at this time began to establish themselves as a specialised group – particularly in Germany – were to develop in the best possible way a version that was closer to the original of each text.

The use of hermeneutics was not restricted to literary texts; it was also applied to the analysis of mathematical texts. As an example, since the Middle Ages, the known versions of Euclid's *Elements* consisted of 15 books, not 13. Eventually, it was shown that the Books 14 and 15 were later additions: Book 14 being authored by Hypsikles (fl. 175 BCE in Alexandria) and Book 15 by Damaskios (fl. 490 in Athens).

The most emblematic philologist was Friedrich August Wolf (1759–1824). He perfected the methods of hermeneutic analysis of classical Greek texts, and systematised his research in relation to the authorship of Homer's two seminal epics, the Iliad and the Odyssey. The classic definition of hermeneutics is due to Wolf: Hermeneutics teaches how to understand and explain the thoughts of an Other through its signs (Wolf 1839, p. 272)[2]:

> Hermeneutics or the art of explaining teaches us to understand the thoughts of an Other by means of their signs and to explain them. This affords the gift of a clear judgement which can penetrate into the analogy of the Other's modes of reasoning […]. What is first of all necessary as scientific knowledge is the knowledge of the language in which the author has written (ibid.).

But for understanding and explaining one needs much more than knowledge of the language:

> This includes several grammatical investigations so that these have to be undertaken first. Knowledge of language, however, will not be sufficient. We must be instructed about the moral situation in his time, we must know about history and literature and we have to know about the mental situation, the spirit [*Geist*] (ibid., p. 273).

Understanding Greek texts, being them epics or inscriptions, requires investigation of the social history as well as the political history of Greece. It is quite revealing that one of the major works of Wolf's disciple August Böckh (1785–1867) was *Die Staatshaushaltung der Athener* (1817; the Political Economy of the Athenians). This means that in the classical understanding of hermeneutics, socio-cultural analyses, analyses of the relevant contexts, had already been conceived. It was Friedrich Daniel Schleiermacher (1768–1834), the great theologian, philosopher, and

[2] Die Hermeneutik oder Erklärungskunst lehrt uns, die Gedanken eines Andern aus ihren Zeichen zu verstehen und zu erklären

philologist, and Wolf's colleague at the new Berlin University, who established hermeneutics as a philosophical discipline. In fact, hermeneutics has since become a specialty of German philosophers. Schleiermacher emphasised, in the same sense as Wolf, as a general concept of hermeneutics and reiterating Wolf, to reproduce what the mind of the Other would have produced. He also stressed that the task of interpretation would be to reconstruct the thought of the Other: One of the main aspects of interpretation is that one must be able to leave one's own thinking and enter that of the writer (Schleiermacher 1959, p. 32).[3]

1.3.2.2 A Subjectivist Variant: Dilthey, Heidegger, Gadamer

By the end of the nineteenth century, however, the German philosopher Wilhelm Dilthey (1833–1911) introduced a deviating conception of hermeneutics. Preoccupied by the dominating socio-cultural position of the exact sciences in the Germany of the second half of the nineteenth century, he aimed at establishing a rivalling strong position for the humanities – *Geisteswissenschaften*. In order to achieve that, he wanted to establish a method that would assume an analogously strong role: a variant of hermeneutics within humanities. Interpreting Schleiermacher's later works – and not knowing all of Schleiermacher's work on hermeneutics –, the task was no longer to get as close as possible to the thought of the Other, but to carry out one's own reconstruction, a subjective reconstruction. The key term for this aspect became "Verstehen", with the meaning of "Einfühlen" – that is to say, to understand, with the sense of empathy.

This 'subjective hermeneutics' was adopted and developed by two more German philosophers, namely Martin Heidegger (1889–1976) – whose anti-rationalistic philosophy led him to support the Nazis – and Hans-Georg Gadamer (1900–2002), likewise opposed to rationalism and to the Enlightenment. Extending Heidegger's approach, Gadamer emphasised to be "freed from the ontological obstructions of the scientific concept of objectivity" (Gadamer 2004, p. 268), and denounced the fight of Enlightenment against "préjugés"[4]; rather, he defined prejudices as the starting point for understanding, calling them the person's "horizon": "a hermeneutical situation is determined by the prejudices that we bring with us. They constitute, then, the horizon of a particular present" (ibid., p. 304 f.). Relating this horizon with "encountering the past and in understanding the tradition from which we come" (ibid.) is called a "fusion of horizons":

> Hence the horizon of the present cannot be formed without the past. There is no more an isolated horizon of the present in itself than there are historical horizons which have to be acquired. Rather, understanding is always the fusion of these horizons supposedly existing by themselves (ibid.).

[3] Eine Hauptsache beim Interpretiren ist, daß man im Stande sein muß aus seiner eignen Gesinnung herauszugehen in die des Schriftstellers.

[4] The overcoming of all prejudices, this global demand of the Enlightenment, will itself prove to be a prejudice (Gadamer 2004, p. 277).

These are individual fusions – intersubjectivity is not addressed. Dilthey's human sciences approach became intensified and popularised so much by Heidegger and Gadamer that nowadays hermeneutics is almost identified with their conceptions, constituting "subjective" hermeneutics (Schubring 2005, pp. 3–5).

While the subjective variant is more used in the analysis of philosophical and literary texts, there are also attempts to use it in mathematics education. For instance, Hans Niels Jahnke proposed a "hermeneutic approach", based on Gadamer, by reading original texts in the classroom and practicing the approach of fusion of horizons, called by him "horizon merging" (Jahnke 2014, p. 75). The aim was to understand a text by Johann Bernoulli. The subject of mathematics was first taught in its modern form and in a modern perspective. For students to read the original text (actually already translated into German), they should apply the following procedures:

– The historical peculiarity of the source is kept as far as possible.
– Students are encouraged to produce free associations.
– The teacher insists on reasoned arguments but not on accepting an interpretation which has to be shared by the entire class.
– The historical understanding of a concept is contrasted with the modern view (Jahnke 2014, p. 84).

Thus, what was proposed to each student was an activity of creating a proper interpretation, not necessarily a historical interpretation shared by all. Moreover, it should only be related to the modern understanding of the concept. As Adriano Dematté commented on this vision of hermeneutics as a fusion of horizons, there is no demand "to reach a homogeneous level for all students" (Dématte 2015, p. 339).

1.3.2.3 Questioning this Variant in Science

Inconsistencies arising by working with such a subjective variant have been addressed in a revealing discussion trying to apply it to science education. The journal *Science & Education* launched from its beginning in 1992 a debate about the use of hermeneutics. It led to the first international *Conference on Hermeneutics and Science*, in 1994, and the founding of the *International Society of Hermeneutics and Science*. One of the contributions within the concept of these debates directly addressed the analysis of textbooks via the conception of hermeneutics. The main focus of the two authors, Fabio Bevilacqua and Enrico Giannetto, in their paper "Hermeneutics and Science Education: The Role of History of Science" was on physics but aimed to be general for the sciences. They were aware of the break caused by the subjective variant, identified by them with Heidegger: "before Heidegger hermeneutics was essentially hermeneutics of texts, interpretation of texts" (Bevilacqua and Giannetto 1995, p. 117). They coin Heidegger's approach as focusing on "lifeworld", and state that in Heidegger's conception lifeworld is opposed to "scienceworld" (ibid.). They comment the relation: "we do not want to enter into the problem of the appraisal of science in the perspective of Heidegger and Gadamer (they give a negative judgement of science)" (ibid., p. 118). They assessed that the subjective variant analyses only semantic aspects: "Mathematical and experimental media are very different from a common language directly related to lifeworld experience"; therefore, they declared the Heidegger & Co. approach as not applicable: "theoretical mathematical language is qualitatively different from natural language, being not directly related to lifeworld experience: they cannot lead us to an understanding of the lifeworld but to the construction of a scienceworld" (ibid., p. 119). According to them, it is just the historical analysis which excludes the subjective approach, upon proposing "a hermeneutical approach to science and history of science that offers an open but not subjectivist interpretation of nature and a view of science that stresses the historical nature of all science texts" (ibid., p. 122).

The proper approach of the two authors to analyse science textbooks follows, then, Kuhn's judgements – but in a surprisingly even more radical manner. On the one hand, they follow Kuhn's separation between textbooks and research papers: "Science has an unavoidable historical dimension; original papers and advanced textbooks are the real depositaries of scientific research". But normal textbooks are entirely dismissed: "Standard textbooks are a caricature not worth it of a hermeneutical analysis" (ibid., p. 115). Only what they call "advanced textbooks" are classified as "more interesting" – being addressed to graduate students instead of undergraduates (ibid, p. 119). They resume, therefore: "A hermeneutical approach to science education in this perspective does not include textbooks because they are mainly related to technical purposes, and stresses the role of original papers and advanced textbooks" (ibid., p. 120).

1.3.2.4 Material or Objective Hermeneutics

However, subjective hermeneutics is not the only practice of hermeneutics at present. At the same time, there is also a strand of hermeneutics that maintains the tradition of classical hermeneutics in the sense of Wolf and Schleiermacher, named objective or material and that insists on the task of getting as close as possible to the thought of the Other. Since the 1960s, philologists Peter Szondi (1929–1971) and Jean Bollack (1923–2012) have strongly criticised the "humanist" philosophical version of hermeneutics and have taken up Spinoza's radical ideas. They developed a "material" version of hermeneutics, which recognises the "otherness" of historical texts and tries to make them objectively legible by placing them in the exact space and time of their production, while striving to reduce the subjectivity of the reader as much as possible (see Bollack 1989). Bollack did not even leave classical Greek literature without criticism of the "subjectivists"; he elaborated new critical interpretations of classical dramas – as opposed to known performances –, such as those of Antigone and Oedipus.

1.3.2.5 A Brazilian Blossom

That the socio-cultural context is an inevitable dimension of textbook analysis is shown, additionally by an instructive example from Brazil, where a rather unique blossom began to develop a short time ago due to a telling point of start. It was the master thesis of Vicente Garnica, who became a leading Brazilian mathematics educator, with strong philosophical interests. In his thesis (Garnica 1992), he had studied whether hermeneutics can be applied to studying mathematical texts, regarding the practice of a mathematics teacher. The conceptual basis there had been Heidegger's notions of hermeneutics. In later studies, Garnica became increasingly dissatisfied with this basis, in particular when he intended to investigate historical mathematical texts. Having also studied the works of Paul Ricœur (1913–2005) – who had been inspired by German philosophers – with plenty of time to read as a war prisoner in Pomerania from 1940, and inspired by Dilthey and Heidegger, Garnica met books by the British sociologist John B. Thompson (b. 1951). Thompson had been inspired by Ricœur, too, but his work was toward an analysis of the phenomena of modern mass media under the aspect of ideology. He called these cultural phenomena "symbolic forms" and searched for a method to analyse them. Having studied the works of Ricœur, he started from a Heideggerian hermeneutics but was unsatisfied, like Garnica, since he understood that investigating the new cultural phenomena demanded "*hermeneutic conditions of social-historical inquiry*" (Thompson 1981, 21). Thompson's efforts were directed to elaborate the social-historical categories he had remarked to need and which were missing in that subjectivist de-contextualised version of hermeneutics. To hermeneutically study these symbolic forms by a socio-cultural analysis, Thompson baptised his no longer subjectivist notion of hermeneutics as *Critical Hermeneutics* (Thompson 1981) and as *Depth Hermeneutics* (Thompson 1981, p. 21).

In a paper written in 2008, together with Fabio Donizeti de Oliveira, a student elaborating a Master's dissertation, Garnica exposed in a careful discussion how to adapt Thomson's socio-cultural analysis of symbolic forms in modern society – not being aware of the material notion of hermeneutics – back to a notion of hermeneutics which would allow him to analyse historical textbooks. He showed to understand that a basic requirement for a hermeneutical analysis has to strive for the originally intended meaning, in terms that correspond well to Wolf's:

> The great methodological discussion about the possibilities of analysis of symbolic forms is to provide an interpretation that is "as close as possible" to what the interpreter understands to be the author's intention, presenting arguments that guarantee that it is the most plausible among the possible ones (Garnica and de Oliveira 2008, p. 37; my transl.).

Garnica extensively adapted Thompson's proposals for a contextual and socio-historical analysis of symbolic forms back to such an analysis of textbooks:

> For this reason, any analysis that is intended to be plausible must consider the contexts of production – the influences that caused the author to produce that and not another work – and the appropriation of symbolic forms. […]
>
> Thus, for example, textbooks are produced to serve a variety of interests, such as those of publishers, those of new educational theories, those of the audiences to which they are

> aimed, educational policies, etc., and an analysis that neglects these contexts, according to the guidelines indicated by Thompson, will turn out to be incomplete (ibid., p. 38).

Garnica had mentioned Thompson's term for his re-invested hermeneutics, of "depth", in this paper only marginally and commented it shortly in a footnote, without any emphasis (ibid. p. 41). His Master's student Oliveira turned the term, however, later on into a programmatic device for the analysis of historical textbooks, without being aware that this was a too complicated deviation for returning to the original, objective or material meaning of hermeneutics. Several students of his group have also taken up this term, without adding new elements enriching the categories of analysis (Schubring 2018a).

1.4 Main Dimensions of Textbook Analysis

Given these methodological discussions as guidelines for textbook analyses, we can now pass to already concretising approaches for the intended contextualised investigations. In a paper written in 1987, devoted to the great textbook entrepreneur Sylvestre-François Lacroix (1765–1843), I had proposed a three-dimensional scheme for analysing the oeuvre of a historical textbook author:

- the first dimension consists in analysing the changes within the various editions of one textbook chosen as starting-point, say an algebra textbook or an arithmetic one;
- the next dimension consists in finding corresponding changes in other textbooks belonging to the same *oeuvre,* by studying those parts dealing with related conceptual fields, say geometric algebra, trigonometry, etc.;
- the third dimension relates the changes within the textbooks to changes in the context: changes in the syllabus, ministerial decrees, didactical debates, evolution of mathematics, changes in epistemology, etc. (Schubring 1987, p. 45).

As one can see, this three-dimensional scheme corresponds well to the likewise three meanings of context, pointed out above: set of values in the society, situation, and meaningful textual system.

Two aspects of contextual analysis shall already be mentioned here, since they will be revealed by many cases in history.

1.4.1 Institution as Co-author

The study of textbooks calls for rethinking the notion of "author". Even for schoolbooks, one uses to accept the name(s) indicated on the title page as the author(s) of the book. Actually, "the" author of a textbook is usually a much larger group than indicated by the names on the title page; the name(s) represents an expanded "collectivity" of contributors. Such a collectivity is a consequence of the fact that textbooks have been linked, at least since the end of the eighteenth century, to an institutional context, and have thus been shaped by the social demands and restrictions of the institution in question: as concerns their syllabi, their typologies of knowledge, and their traditions. Thus, institutions should be considered a determining factor for the collectivity of textbook authors. How decisive these institutional and collective factors are can also be illustrated by the large number of books published without any indication of authorship.

It is likewise only rarely analysed as a characteristic of textbook production which position or rank the authors hold in the community at large. There are marked differences – over the centuries of textbook production within one country and comparing different countries as well. Focussing for instance on textbooks for secondary schools, one can identify teachers of secondary schools themselves, or leading authors from higher education types like pedagogical colleges, or even – as until quite recently in the Soviet Union – Academicians. A promising methodology for revealing profiles of textbook authors as well as differences and changes in profiles is prosopography, which investigates collective biographies of determined social groups (see Shapin and Thackray 1974; Kouamé 2015).

1.4.2 *Textbook Knowledge as Common Property*

In the analysis of Lacroix as a textbook entrepreneur, I had also pointed to a consequence of a collectivity of authors behind the official author (Schubring 1987): The collectivity of authorship is also shown by the fact that a textbook is in general moulded in contents and structure by the already existing textbooks for the particular institution, and by frequent "borrowings" from other books, or even by direct copying.

This is an expression of a particularly remarkable pattern: school knowledge is regarded, unlike research knowledge, as a sort of "common" property. Copyright regulations and respect for the rights of authors find practically no application in this field of publication. Lacroix presents a telling illustration of this particular character of schoolbook knowledge: In the first two editions of his algebra textbook, he had no problem to tell of having "borrowed" from an earlier textbook by Bézout some "details", "which are common to all books and all methods" (Schubring 1987, p. 45; see chap. 6). This character as common property reinforces even more the necessity of refined methodological approaches.

Chapter 2
Textbooks Before the Invention of the Printing Press: Orality and Teaching

2.1 The Poles of Orality and Writing

Commonly one might associate textbooks as evidently being bound to printed forms – to books, as their name already seems to imply. Considering the history of teaching, it becomes clear, however, that textbooks were used in teaching since Antiquity, in different cultures. Textbooks existed, hence, in some manuscript form. And not being reproducible, one understands that the notion of textbook does not depend on the possibility that each student has a personal copy. This reveals an additional basic pattern for the use of textbooks as well as for teaching in the epochs before the technology of printing: it is the orality as the dominant form of teaching – a pattern rarely ever considered for conceptions of textbook analysis.

All cultures based on script, sooner or later, developed a technology for preserving teaching materials on more or less durable media. The constraints imposed on the dissemination of books before the invention of paper were the fact that the materials to write upon were rare and expensive, like parchment in Europe, or difficult to handle and preserve, like clay tablets for cuneiform texts in Mesopotamia, papyrus in Egypt, dried palm leaves in India, etc.

Those texts were so difficult to reproduce that it required teachers to rely on methods other than having students read written documents for transmitting knowledge. Yet, the very complexity of the early techniques for writing and reckoning required that they be taught systematically according to standardised methods. All cultures disposing of a script of their own somewhen began to standardise and institutionalise their teaching for young people. This led them to develop a certain corpus of knowledge, and to formalise this body of knowledge into a canon. However, this standardised type of knowledge was conveyed mainly via oral teaching. Eventually, the existing written versions of that standard knowledge were used merely as memorisation aid.

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
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Moreover, there were crucial cultural determinants of this oral form of teaching. The entire cultural sphere of ancient civilisations was based on orality, and the ability to memorise even very long texts was highly valued and emphasised. The preeminence of orality dominated all cultures until Modern Times, and the art of memorisation fell into disrespect only a few generations ago.

Not only these historical reasons, but also more general reasons make textbooks inseparable from teaching. It is no coincidence that the self-learner is widely sneered at, and the standard epithet "autodidact" for a self-educated person conveys the impression that he or she has failed to grasp essential dimensions of the subject in question. The belief behind this is that implicit notions or underlying values cannot be assimilated from written or printed matter, as they are only understandable from social interaction with "scholars". In this way, learning, even under the aspect of using textbooks, always implies a *three-pole* relationship, between the *teacher* (or professor), representing orality, and the *textbook*, representing the script, and the *student*, representing the receiver.

In this relationship, orality was the dominant pole in all cultures that existed before the invention of the printing press. As it was possible to produce written texts, but difficult to reproduce them, the primary goal and teaching method consisted in having the students to memorise texts. The teacher's main task, therefore, was to assure the fidelity of text transmission and to prevent students from introducing changes or errors. It is evident that this education system was hostile to innovation, due to these structural reasons. It was static and led to structural separation between "research" and teaching, since teachers did not need to qualify as scholars, and they were not even required to be experts in the subjects they taught. The students, on the other hand, were kept to remain passive and receptive – their task being to adhere unconditionally to the traditional ways and methods, to learn by rote and memorisation, and to reproduce the contents thus impressed on their minds with unfailing precision.

It is necessary to emphasise another fundamental aspect of text use in this period of development before the invention of the printing press. One more compelling reason for the scarcity of written textbooks is that most of the early transmission of knowledge to the next generation was ensured by professional guilds in the service of the central administration of the respective state. To enhance and safeguard the eminence and holiness of the early states, knowledge and the art of writing were kept arcane and inaccessible to the general population. This arcane and guild-specific character of scripture preceded the emergence of any need for technologies or techniques that would allow for more reproducibility of texts.

A very revealing proof for such structures is an inscription on the stele of an Egyptian artisan and scribe (around 2000 BCE):

> I know the secrets of the hieroglyphs and of the procedures for the ritual of the feasts […]. I will uncover this knowledge to nobody, except to my eldest son; the divine sovereign has authorised me to uncover the secrets to him.[1]

Even afterwards, the practice of keeping knowledge secret continued; let us remember, for example, the Pythagorean school of philosophy and mathematics, and the architects of the medieval cathedrals who kept a great secret in relation to their applied geometric knowledge.

In the history of textbooks, the following two dimensions have proved to be underlying structural patterns of development:

- The process of transforming the secret and guild-specific knowledge into general and public knowledge.
- The relationship between orality and literality, and, closely related to this aspect, the relationship between research and teaching, and between receptivity and activity, in the teaching-learning process.

As the analyses in the chapters of this book will reveal, the three poles were not configured in a stable, uniform, and permanent manner; rather, history shows a great variability in the relationships between the poles. A first overview of different configurations will be presented here.

Figure 2.1 (below) shows the relationship between the three poles as characteristic for Antiquity and for Medieval Times – as will be discussed in this chapter, only few textbooks were elaborated for teaching mathematics. The teachers were, hence, not their producers, they were the "organ" of the textbook (a term used later in nineteenth-century Prussia to denounce strict prescriptions of the textbooks by the authorities; see Chap. 8), which was speaking to the – passively receiving - students *via* the teacher. Thus, despite the orality dominating the teaching, it was not the teacher who dominated the construction and organisation of the knowledge to be taught, but the textbook – heir and representative of a conception of knowledge established typically a long time before. This triangle form will prove to have been practiced also at various instances in Modern Times; it became characteristic again during the period of 'new math', in the USA of the 1960s and 1970s when one wanted to elaborate the so-called "teacher-proof" curriculum – that meant to develop such a kind of textbooks that the students would learn successfully with them, regardless a possibly low performance of the teacher (see later chapters).

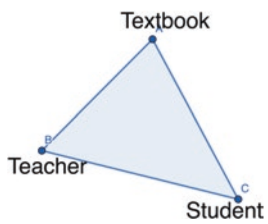

Fig. 2.1 The "characteristic triangle" of textbook use: the three poles

[1] This stele (see Fig. 2.9) was made for Irtisen, the overseer of the artisans, of the period of the king Mentonhotep II (ca. 2060-2010 B.C.E.); Musée du Louvre, Paris.

In some cases, the teacher might act on the same level as the textbook: enriching it with commentaries, or improving what he might assess as an erroneous or unclear proposition. One might symbolise this case by this figure (Fig. 2.2):

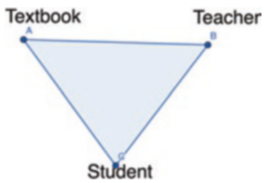

Fig. 2.2 The teacher handling the textbook actively

The characteristic practice where the teacher dominates the teaching process, the textbook being a means handled by a proper mastery of subject and method, is symbolised by Fig. 2.3. It was first realised in nineteenth-century Prussia (see Chap. 8).

Fig. 2.3 The teacher-being-dominant characteristic

A fourth, "degenerated" form has also been practiced in various periods: when the triangle is reduced to a line (Fig. 2.4). This is the case when there is no teacher; and it constitutes the self-learner's situation. Textbooks explicitly claiming to serve – also – for direct learning from the book, without need for a teacher, were published shortly after the first printed mathematics books. A variant of this form appeared when the teacher became himself the student: noteworthy in the case of the first textbooks exposing the infinitesimal calculus. Then the university professors were the first ones to learn the new discipline from textbooks (see Chap. 4).

Fig. 2.4 The degenerate characteristic: teaching and learning without teacher

Evidently, these various characteristics of the relationship between the three main actors in the teaching-learning process prefigure decisively the structure, outline, conception, and organisation of the knowledge to be taught by the textbook. They constitute, therefore, a basic requirement for textbook analysis.

After these general remarks and the additional dimensions for textbook analysis, we can now enter the analysis with textbooks in cultures of Antiquity.

2.2 Mesopotamia

It seems that institutionalised teaching of mathematics first appeared in Mesopotamia. Since the earliest proto-Sumerian era and since the invention of script, even before 3000 BCE, there is ample evidence of institutionalised teaching. Around 2500 BCE, the scribes emerged as a professional group; the institutionalised centre for their training and activities was called *edubba* (House of Tablets). During the classical Babylonian period, called Old-Babylonian (marked by the king Hammurabi, eighteenth century BCE), the scribes' guild had already gained a certain autonomy, which is proved by the fact that they proposed relatively abstract algebra problems (disguised as solving practical problems). While a complete epic poem of the Sumerian period, the Gilgamesch, was found, as far as literature is concerned, until now, no extensive mathematical texts have been discovered, as yet for the Sumerian and Babylonian eras. However, a large number of cuneiform tablets with particular mathematical texts have been preserved, especially from the classical Old-Babylonian period (around 2000 to 1600 BCE). Among them, different types can be distinguished: home exercises and problems for students, and manuals for teacher use (see Ritter 1994, p. 274). Some tablets are even small manuals, like the one with 21 algebra problems (cf. Caveing 1994, p. 21 and 35). Typically, there are ritual texts that provide the scribe with the particular algorithm to be used (Ritter 1989, p. 31).

The approximately 2000 known tablets containing mathematical texts have been extensively studied during the last decades. While one used to classify them as texts produced for educational purposes, modern research is more cautious. While a great number of the tablets deal with elementary mathematics and can be attributed to teaching aims, there are tablets with high-level mathematics which can be assessed to have been produced by erudite scholars, involved in teaching and elaborated in the same place, but not necessarily for teaching aims (Proust 2019, p. 11). In particular, a great part of the higher-level mathematical texts "do not clearly reveal the exact context in which they were composed or used" (Proust 2014, p. 29). The identification of the context is hampered considerably by the fact that the provenances of many tablets are unknown, due to having been bought by grave robbers or not being well documented (ibid., p. 28 f.).

Yet, most of the Old Babylonian mathematical tablets are school tablets and have been mainly found in Nippur, the political and cultural capital of Old Mesopotamia. Historians have succeeded in reconstructing, by analysing these tablets, the mathematical curriculum of the *edubba* in Nippur, maybe analogous to that of *edubba* in other cities. Archaeologists were even able to identify one building as that of an *edubba*: the house F (Robson 2008, p. 98).

A highly noteworthy result of the analyses of the school tablets has been that their shape provides evidence of the level of teaching to which they were destined in the curriculum (Fig. 2.5):

Fig. 2.5 Type II tablet. Obverse: lexical list; reverse: measures of capacities. (Proust 2007, pl. 26; quoted from Proust 2014, p. 31)

> The tablets used most often at Nippur (type II in the typology of Assyriologists) are large rectangular tablets (about 10 cm • 15 cm), which the young apprentices used in their training to memorize and write a set of standardized texts. These texts included lists of cuneiform signs, Sumerian vocabulary, systems of measurement, and elementary numerical tables. When a long series of lexical lists or mathematical tables had been completely memorized, it was written on large multi-column tablets known as "Type I" or, sometimes, on prisms. These great compositions on prisms may be interpreted as a kind of examination (Veldhuis 1997, p. 31; Proust 2014, p. 30).

Type II tablets allowed historians to identify the knowledge taught to young scribes at an early stage of schooling. In particular, Veldhuis was able to show that the reverse of those tablets served as a repetition of a school text studied at a point earlier in the curriculum (Veldhuis 1997, p. 36; quoted by Proust 2014, p. 30). Figure 2.5 shows both sides of a Type II tablet. The method of Veldhuis has been applied by Proust to mathematical texts (Fig. 2.6):

> In addition to these exercises, scribes would sometimes note short excerpts on small single-column rectangular tablets (Type III – see Fig. [2.6]). The Sumerian name of this type of tablet sometimes appears at the end of the composition, as well as in some literary texts: *imgidda* or "elongated tablets". *Imgidda* tablets were often used to learn multiplication tables (as shown in Fig. [2.6]) (ibid.).

It was possible for researchers to systematically relate the forms of tablets to functions in teaching:

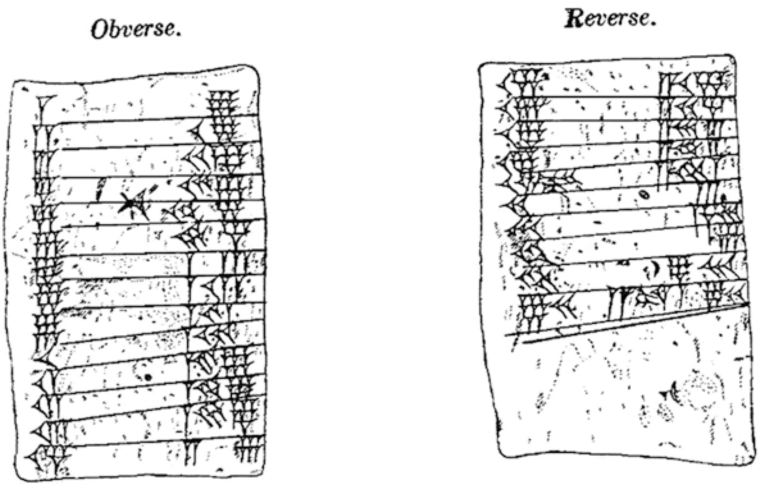

Fig. 2.6 Type III tablet: multiplication table. (Quoted from Bernard and Proust 2014a, b, c, p. 32)

- type I (large multi-column rectangular tablets, the text on the reverse side follows on from that on the front)
- type II (large multi-column rectangular tablets, the text on the reverse side does not follow on from that on the front)
- type III (small single-column rectangular tablets, mainly containing numerical tables, Fig. 2.6) (Proust 2021, p. [1]).

Analysing the texts according to their different formats, it was possible to characterise the curriculum for elementary education in Nippur as constituted by three levels: elementary, intermediate, and advanced (Table 2.1). Its first level was devoted to learning the following lists and tables more or less in this order: lists enumerating measurements of capacity, weight, surface, and length; tables providing correspondence between the various measures and numbers written in sexagesimal place value notation (abbreviated: SPVN); and numerical tables (tables of reciprocals, multiplication, squares, square roots, and cube roots). All of these elementary lists were probably learned by rote (ibid., p. 31). The knowledge taught at the intermediate level could also be identified in the texts of the cuneiform tablets:

Table 2.1 The first two levels of the mathematical curriculum in Nippur (Bernard and Proust 2014a, b, c, p. 33). The figures mentioned in this table refer to the chapter quoted here

Level	Content	Typology	Examples
Elementary	Metrological lists (capacities, weights, surfaces, lengths) Metrological tables Numerical tables (reciprocals, multiplication, squares) Square and cube roots	Types I, II and III	See Figs. 2.3 and 2.4
Intermediate	Exercises: multiplications and reciprocals. Surface and volume calculations	Square-shaped tablets	See Robson (2001, p. 98)

> At this less formalized level, scribes learned the basics of sexagesimal calculation, namely multiplication and calculation of the reciprocal of large numbers.
>
> This knowledge was then applied to finding areas and volumes. At schools in Nippur, this level of education is documented mainly through exercises noted on square-shaped tablets (ibid., p. 32).

Recent research has focused on Type IV tablets, as tablets serving intermediate teaching used to be called; a great part of them stem from Nippur: “42 type IV mathematical tablets from Nippur have so far been identified. Most of them are square; a few are lenticular” (Proust 2021, p. [4]). These forms have been understood, due to their layout and shape, as “scratch pad for a problem text”, to “be held in the palm of the hand” (ibid., p. [11]). These tablets form a remarkably “homogeneous whole”, containing a “consistent set of exercises”: “the set of exercises from Nippur reflect an elaborate pedagogical project” (ibid.). And their type of exercises prepared for advanced learning levels (ibid., p. [14]).

Different from the elementary and intermediate level ones, texts for advanced learning are not easy to reconstruct in particular to distinguish texts aimed for teaching from others reflecting scholarship. The criterion of complexity of the mathematical procedures might easily be subject to anachronism. As more reliable, material aspects have been suggested. Various cases of single column tablets (Type S) could be attributed to advanced level teaching. One such tablet is reproduced here (Fig. 2.7):

Fig. 2.7 Type S tablet: YBC 4663; with kind permission by the Yale Babylonian Collection (New Haven, Connecticut). (An image where all the sides of this tablet are shown is accessible at: https://collections.peabody.yale.edu/search/Record/YPM-BC-018728)

> This tablet has an elongated shape and is written in a single column (type S). The tablet is of unknown origin, but probably comes from a city in southern Mesopotamia. It contains a sequence of eight solved problems dealing with digging trenches. The parameters of the problem (data and unknowns) are the dimensions of the trench (length, width, depth), its base, the volume of extracted earth, the number of workers needed for digging, the daily labor assigned the workers (namely, the volume of earth to be extracted each day by each worker), their daily wage, and the total wages (expressed as a weight of silver) (Proust 2014, p. 33f.).

Proust has given a highly telling assessment of the levels of the curriculum and the sequence of mathematical operations and abilities to handle them, as evidenced by the analysis of cuneiform tablets found in scribal schools of Southern Mesopotamia:

> In the elementary level the basic tools of calculation on numbers in sexagesimal place-value notation are introduced through multiplication and reciprocal tables. Moreover, the correspondence between these numbers in SPVN [Sexagesimal Place-Value Notation] and the measurements of capacity, weight, length and surface is fixed through the metrological tables. Intermediate level exercises focus on solving simple linear problems, i.e. involving short sequences of multiplications and reciprocals. These exercises are based on the mastering of elementary numerical and metrological tables. The catalogue exercises, in the first part of the cycles, also deal with linear problems, starting with a problem similar to the exercises found in type IV tablets, then using more abstract conceptions to multiplication and reciprocal. This set of mathematical tables, exercises and problems, belonging to elementary and intermediate levels, as well as to the first part of cycles, forms a strongly coherent mathematical corpus that can be described as a "linear paradigm" (more on that in Proust 2019). It is only at a more advanced stage, reflected by the second part of the catalogue cycles, that quadratic problems appear, and with them a novelty: the addition and subtraction of numbers in SPVN. Because of their floating nature, these numbers are not suitable for addition and subtraction, which explains why these operations are absent from the lower levels (Proust 2021, p. 17 f.).

The famous Plimpton 322 tablet (Fig. 2.8), regarded by Jöran Friberg and Eleanor Robson as intended for teaching, is not clearly linked to teaching, according to a new detailed study by Proust and John P. Britton, and Steve Shnider (Britton et al. 2011).

Fig. 2.8 The plimpton 322 tablet. (Britton et al. 2011, p. 564)

2.3 Egypt

Unfortunately, and contrary to Mesopotamia, only few mathematical manuscripts are preserved from Egypt, despite its several millennia of mathematical practices. This is basically due to the fact that papyri need absolute dryness not to be destroyed – it was, therefore, better preserved in desert regions of Egypt. Annette Imhausen has assembled for her *Contextual History* of Egyptian mathematics an enormous number of sources – also beyond papyri, such as inscriptions on tombs and steles – to analyse the texts and their contexts.

Likewise, one has no direct evidence for the existence of an instutionalised system of scribal education. During the Old Kingdom, this seems to have happened within the family, the father instructing the son as an apprentice (see the Irtisen quote in Chap. 1; Fig. 2.9). For the Middle Kingdom, there are some hints to organised instruction of scribes but no clear-cut evidence. There are no mathematical texts preserved from the Old Kingdom. All the rare and famous manuscripts date from the Middle Kingdom. They are written in hieratic script. There have been doubts about the teaching function of these texts (Ross 2014, p. 37), but Imhausen affirms that:

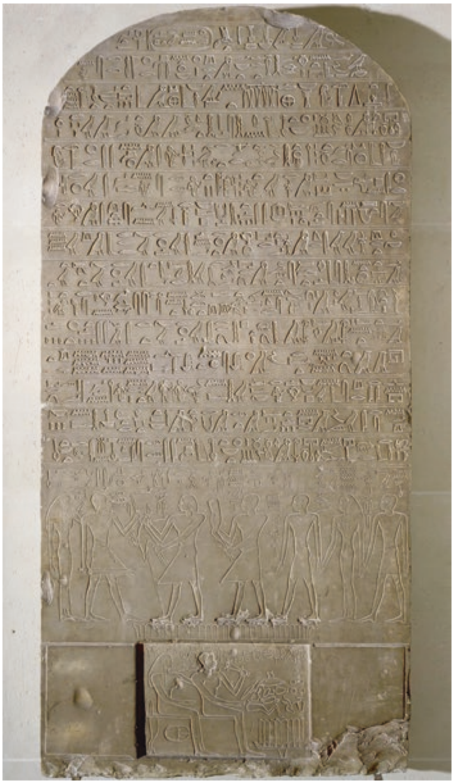

Fig. 2.9 Irtisen stela, Musée du Louvre

> The majority of the hieratic mathematical texts come from an educational context. The largest extant source, the Rhind mathematical papyrus, is likely to have served as the manual of a teacher. The content of Egyptian mathematical texts can be divided into two categories: procedure texts (or problem texts) and table texts. Procedure texts present a mathematical problem, followed by instructions for its solution. Table texts are tabular arrangements of numbers used as aids in calculations. Extant table texts from Egypt include tables for fraction reckoning as well as tables for the conversion of measures. A single source may contain one table or problem only or present a collection of tables and problem texts (Imhausen 2016, p. 64f.).

The original of the Rhind papyrus (Fig. 2.10) was written during the Second Intermediate Period, around 1550 BCE, but a part of it is copied from a text from the end of the Middle Kingdom. The edition mostly used today is that by Thomas Eric Peet (1923). The title of the papyrus can be translated as "Rules for inquiring into nature, and for knowing all that exists, [every] mystery … every secret" (ibid., p. 66). It consists of 64 problems and a number of tables – the numeration (up to 87) was introduced by August Eisenlohr in his edition of 1877. Imhausen characterises its contents as follows:

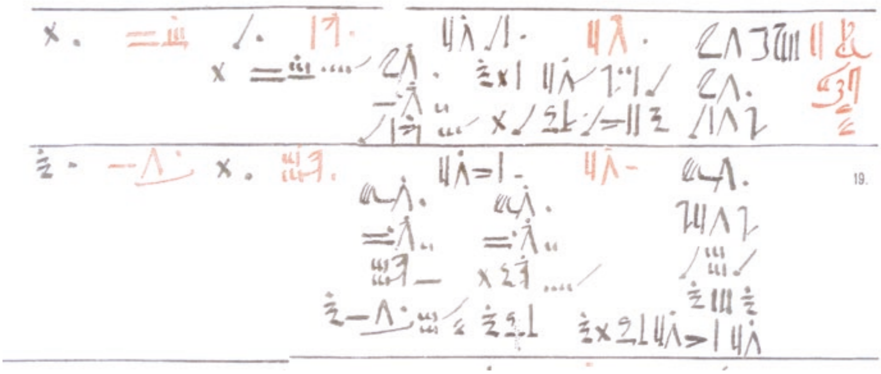

Fig. 2.10 Problems 17 and 19 from the Rhind papyrus, treating divisions. (Eisenlohr 1877, table II)

> However, numbers 7–20 are mere calculations, which are more likely to be associated with the creation of a table, as are the calculations found as number 61. Numbers 80 and 81, likewise, are better referred to as table texts, while 82–84 are (model) accounts. Following these are: in number 85 an unintelligible group of signs, in 86, a fragment of accounts, and in 87, calendrical entries. Two further problems, 59B and 61B, have to be added to the count (ibid., p. 67).

It is very interesting that the problems do not only contain instructions to find the solution, but in some cases they also indicate how to verify what was found:

> In addition to these fully written calculations, some problems include notices and numbers that can best be described as notes during the (mental?) performance of a calculation. These

> are mostly found after the instructions. Once the final solution is obtained (by having followed the instructions), a verification of the solution may be carried out. The verification can consist of a set of instructions, written calculations, or both (ibid., p. 77).

The second most important and large text is the Moscow papyrus, from around 1800 BCE, exposing 25 problems:

> It includes the two most intriguing problems of Egyptian mathematics, the calculation of the volume of a truncated pyramid (number 14) and the calculation of a *nb.t* (number 10). This has been interpreted (among other possibilities) as calculation of the surface of a half sphere or the surface of a half cylinder (ibid., p. 69).

To study the practice of teaching, one should understand the structure of procedure texts besides the tables, the principal form of Egyptian mathematics known to us. Imhausen has investigated extensively this format:

> A procedure text begins by stating a mathematical problem. After the type of problem is announced, some specific data in the form of numerical values are given, thus specifying the problem to one concrete instance or object. This is followed by instructions (the procedure) for its solution. The title, specifications of the problem, and the following instructions are expressed in prose, using no mathematical symbolism. The title (and other parts of the text) may be accentuated by the use of red ink, which is rendered in the translation as small caps.
>
> Each instruction usually consists of one arithmetic operation (addition, subtraction, multiplication, division, halving, squaring, extraction of the square root, calculation of the inverse of a number) and the result of it is given. The instructions always use the specific numerical values assigned to the problem. Abstract formulas, or equations with variables, did not exist (ibid., p. 70).

She alerted to the risk of anachronism in modernising Egyptian mathematical procedures to present-day procedures: Many earlier studies of Egyptian mathematics focused on this type of problem; it is one of the examples where a "modern mathematician" has the impression of knowing at first glance (given that he or she is presented with a translation into a modern language) what is going on "mathematically." She remembered the controversy whether some of the problems were solved by the so-called method of false position, "while others took them to be evidence that the Egyptian procedure is 'equivalent' to our modern manipulation of algebraic equations" (ibid., p. 71).

A procedure text may contain drawings, "usually placed at the end of a problem, after the section of instructions. They are not technical drawings in the modern sense, but rather rough sketches illustrating the problem" (ibid., p. 76). There have been expectations that ancient scribes would have drawn "proper sketches", and thus one has measured the sketches to derive mathematical techniques from them; however, "Egyptian drawings are not to scale" so that such interpretations are anachronistic (ibid.). It is revealing to encounter here a problem we will meet again, more extensively, with the issue of diagrams accompanying Greek mathematical texts (see Sect. 2.4).

2.4 China

China is known as a highly bureaucratic state, and was the first nation to introduce sophisticated official examinations to regulate access to administrative careers. Mathematics, although not much appreciated by the dominant religion, Confucianism, was part of these exams. Textbooks were key elements of teaching, as stipulated by the decreed curriculum. Likewise novel is that, even before this introduction of a curriculum based on textbooks, various mathematical textbooks have been elaborated.

In the last decades, several arithmetic manuscripts have been found dating from the second half of the first millennium BCE, but more detailed knowledge on mathematics in China is only available from the first unification of China under the Emperor Shi-Haung-ti (221–207 BCE) and the subsequent Han dynasty (206 BCE to 220 CE). This development began with the writing of textbooks: the textbook *Jiu Zhang Suan Shu*, by an anonymous author, is the emblematic work for teaching mathematics in China and played an analogous role for Asia as Euclid's *Elements* did for Europe; commonly, it is mostly dated around 200 BCE. They had been written first on rolls of different materials, especially on bamboo slats. Subsequently, writing and using them became easier due to the Chinese invention of paper. An excellent overall account of the history of mathematics in China is Martzloff's book, first published in French in 1987 and updated in English in 2006.

In general, it is a characteristic of China that the development of culture and especially of mathematics is more closely linked to political history than it is in other cultures: due to considerable discontinuities in internal wars and invasions. During a pillage in 1127 CE in Kaifeng, then capital of China, all copies and printing blocks of mathematical books had been destroyed; thereafter, an official undertook the search for extant copies throughout the provinces and was able to find most of the books and have them reprinted. However, one book was found only incomplete (Volkov 2014, p. 66 f.).

The title of *Jiu Zhang Suan Shu* has been translated in different ways: *Computational Prescriptions in Nine Chapters* (Martzloff), *Computational Procedures of Nine Categories* (Volkov), *Les neuf chapitres sur les procédures mathématiques* (Chemla and Shuchun 2004); Joseph W. Dauben, in his evaluation of the numerous proposed translations, argued for *Nine Categories of Mathematical Methods* as the best adapted translation (Dauben 2013, p. 204). I am preferring here *Nine Chapters of Mathematical Practices*. The book was further developed by numerous commentaries in the first millennium. The most important is that by Liu Hui (ca. 220–290) and later by Li Chufeng (656). Both had been "systematically interested in establishing the correctness of the algorithms set out in The Nine Chapters. They are the ones who, in relation to this last activity, introduce different types of visual aids, which remain totally absent from the Classic" (Chemla and Shuchun 2004, p. 5).

There have been several recent translations. An unabridged translation into English of the complete text, together with the entire set of commentaries, was published in 1999 by Shen Kangshen, John C. Crossley, and Anthony W.-C. Lun. It is based on a 1964 edition of the ancient text that had been edited in modern Chinese. A meticulous translation, based on the original text, with commentaries on the text and the contemporary commentaries, was published in 2004 by Karine Chemla and Gua Shuchun, as a bilingual Chinese-French translation. Dauben, in his critical analysis of translations into various languages, emphasises the enormous methodological and technical merits of their translation (Dauben 2013, pp. 220 f.). Dauben himself, together with Xu Yibao, has published a critical English translation in three volumes, based on Guo Shuchun's rendering of the ancient text and its modern Chinese translation (Dauben et al. 2013).

The book is constituted by a collection of 246 three-part problems, which always contain: the formulation of the problem, the numerical answer, and a guide to calculate the solution from the data. The contents of the *Nine Chapters* can be characterised as follows:

1. Measurement of fields: calculating the area of rectangles, circles, circle segments. Calculating with fractions.
2. Exchange of crops by converting grain and field crops into units of millet. Rule of three.
3. Compensation of tax units, by money or labour power, between different regions.
4. Geometric problems, e.g. determining the side of a square or cube.
5. Determination of work performances for the execution of various construction works, calculating the volume of prisms, pyramids, tetrahedra and truncated cones.
6. Calculations for the transport of goods, determination of the number of soldiers depending on the population of the region concerned.
7. Methods of solving linear equations, using the double false approach method.
8. This chapter, named *fangcheng* (square arrangement), is valued as the mathematically most significant one. It shows solving problems by manipulating numbers arranged in table form in parallel columns. It corresponds to the solution of linear systems in n unknowns (≤ 6), which has been interpreted as using matrices.[2] This chapter presents major challenges to understand and translate it (see Table 2.2). For instance, regarding what one can understand as a rule of signs, Chemla & Guo Shuchun translate modernising a term which commentator Liu Hui gives as quantities (gains and losses represented by red and black counting rods), as "numbers" (*Nombres de même nom…*) (Chemla and Guo Shuchun 2004, p. 627), while Dauben & Yibao just refer to quantities ("If the signs are the same …"), (Dauben et al. 2013, vol. 1, p. 248).

[2] At the same time, the attribution of negative numbers to this Chinese teatise is based on this chapter (see Schubring 2005, p. 36).

Table 2.2 The challenge of translating, implying interpretation

同名相除 *tong ming xiang chu*	SI DES NOMBRES DE MÊME NOM SONT ÉLIMINÉS L'UN DE L'AUTRE (Chemla & Guo Shuchun 2004, p. 627). IF THE SIGNS ARE THE SAME (同名 TONG MING), THEN SUBTRACT ONE FROM THE OTHER (Dauben et al. 2013, p. 248).

An original text from Chap. 8 (Chemla and Guo Shuchun 2004, p. 624), and a French and an English translation (I am thanking Joseph W. Dauben in assisting me to establish the correspondences)

9. Right-angled triangles: application of the Pythagoras "theorem" to determine a side or the diagonal of a square (Martzloff 2006, p. 132 ff.; Wußing 2008, p. 56 ff.).

Usually, the practice of Chinese mathematics has been characterised as presenting only results, but no justifications for the procedures (Martzloff 2006, p. 69). "Argumentative discourses", which one finds in commentaries on the *Nine Chapters* from the first millennium CE, have been presented as an exception (ibid.). Chemla and Shuchun, however, have insisted that the traditional view of the book being "a collection of practical situations for the resolution of that which narrow-minded practitioners would be too happy to find recipes, in *The Nine Chapters*" is not correct, and that one can also find there "demonstrations" for the procedures (Chemla and Shuchun 2004, p. xiv and 33).

> Generally speaking, the problems of the Classic are particular for two reasons: they describe a singular situation, and they propose numerical values for the data. […]
>
> However, some problems do not meet this description, insofar as, while they attribute particular values to the data, the situation in the context of which they are posed is itself abstract. […]
>
> The Nine Chapters therefore combine statements which appear to us in turn concrete, recreational, or abstract (ibid., p. 8f.; my transl.).

Justifications for algorithms are due to commentators of the Classic, in particular to Liu Hui's comments. As shown by Chemla, the comments sought to make unclear passages in the *Nine Chapters* easier to understand and to justify the algorithms (Chemla 2014). At the same time, this implies that the original text does not contain such justifications. In fact, Chemla and Shuchun admit that:

> While *The Nine Chapters* seem not to pay any attention to recording the reasons why the algorithms provided are correct, it is one of the primary objectives of the exegetes, who systematically address this question in commentary to the statement of each procedures (ibid., p. 26; my transl.).

Actually, it constitutes a practice that can be observed also in other cultures that in periods without significant innovations attempts are made to better justify the classical texts through comments, for example, in the case of the reception of Euclid in the Islamic civilisation (see Sect. 2.6).

Although mathematics played only a minor role in the Confucian culture of China, compared with the occupations of the *literati* (Martzloff 2006, p. 79), it had a stable role in professional education. The development of mathematics in China is characterised by an institutionalisation within the training of civil servants – known in Europe as mandarins – in the centrally well-organised state administration. Since the Sui dynasty (518–617) there was a school of mathematics within the *guozixue*, the "school for the sons of the state", which included several areas of civil servant training, such as the departments of Classics, Law, and Medicine; the graduates would mainly work in financial administration (Volkov 2014, p. 58). This so-called *School of Computations* also existed during the Sui Dynasty (581–617), which reunited the previously divided country. A strong development of the school followed in the Tang Dynasty (618–907), from 628 onwards. The year 656 marks a generally significant decision: this was the first time that a curriculum was defined for teaching and examining students – in fact, the first known curriculum for teaching mathematics. The curriculum – for 7-year studies – consisted of the indication of textbooks and each level at which they should be taught. This is the famous list of the *Ten Classics* – which, however, has always consisted of 12 textbooks (Table 2.3). Martzloff described their level as in general "low" (Martzloff 2006, p. 16).

Table 2.3 Conventional identification of the Tang dynasty textbooks with the extant mathematical treatises (Volkov 2014, p. 62)

#	Treatises as listed in the *Jiu Tang shu* and *Xin Tang shu*	The extant treatises with which the Tang dynasty treatises are conventionally identified	Author	Date of compilation of the extant treatise
1	*Sun zi* 孫子 ([*Treatise of*] *Master Sun*)	*Sun zi suan jing* 孫子筭經 (*Computational Treatise of Master Sun*)	Unknown	Ca. AD 400 (?)
2	*Wu cao* 五曹 (*Five Departments*)	*Wu cao suan jing* 五曹筭經 (*Computational Treatise of Five Departments*)	Unknown	Not earlier than 386 AD.
3	*Jiu zhang*九章 (*Nine Categories*)	*Jiu zhang suan shu*九章筭術 (*Computational Procedures of Nine Categories*)	Unknown	Prior to the mid-first century AD
4	*Hai dao* 海島 (*Sea Island*)	*Hai dao suan jing* 海島筭經 (*Computational Treatise* [*Beginning with a Problem about a*] *Sea Island*)	Liu Hui	Ca. AD 263
5	*Zhang Qiujian* 張丘建 ([*Treatise of*] *Zhang Qiujian*)	*Zhang Qiujian suan jing*張丘建筭經 (*Computational Treatise of Zhang Qiujian*)	Zhang Qiujian 張丘建 (dates unknown)	Mid-fifth century AD
6	*Xiahou Yang* 夏侯陽 ([*Treatise of*] *Xiahou Yang*)	*Xiahou Yang suan jing* 夏侯陽筭經 (*Computational Treatise of Xiahou Yang*)	Han Yan 韓延 (dates unknown)	763-779
7	*Zhou bi* 周髀 (*Gnomon of the Zhou* [*Dynasty*])	*Zhou bi suan jing* 周髀筭經 (*Computational Treatise on the Gnomon of the Zhou* [*Dynasty*])	Unknown	Early first century AD (?)

(continued)

Table 2.3 (continued)

#	Treatises as listed in the *Jiu Tang shu* and *Xin Tang shu*	The extant treatises with which the Tang dynasty treatises are conventionally identified	Author	Date of compilation of the extant treatise
8	*Wu jing suan* 五經筭 (*Computations in the Five Classical Books*)	*Wu jing suan shu* 五經筭術 (*Computational Procedures in the Five Classical Books*)	Zhen Luan	ca. AD 570
9	*Zhui shu* 綴術 (*Procedures of Mending* [=*interpolation?*])	*Lost*	Zu Chongzhi 祖沖之 (429–500)	Second half of the fifth century AD
10	*Qi gu* 緝古 (*Continuation* [*of Tradition*] *of Ancient* [*Authors*])	*Qi gu suan jing* 緝古筭經 (*Computational Treatise on the Continuation* [*of Tradition*] *of Ancient* [*authors*])	Wang Xiaotong 王孝通 (b. ?- d. after AD 626)	ca. AD 626
11	*Ji yi* 記遺 (*Records Left Behind for Posterity*)	*Shu shu ji yi* 數術記遺 (*Records of the Procedures of Numbering Left Behind for Posterity*)	Xu Yue 徐岳 (b. before 185 – d. after 227)	ca. AD 220
12	*San deng shu* 三等數 (*Numbers of Three Ranks*)	*Lost*	Dong Quan董泉 (dates unknown)	Prior to AD 570

School students were divided into two groups; the first one, called the "regular" course, studied the first eight textbooks of the curriculum, and the second group, the two "advanced" courses, studied textbooks 9 and 10. The last two textbooks, indicated as "compulsory", should be studied simultaneously with those of the two courses. The textbooks for the regular course should be studied mostly for 1 year, only the *Jiu Shang Suan Shu* for 3 years. The two advanced textbooks should be studied during three, respectively 4 years (ibid., p. 61).

According to more detailed reports, there were always shorter or longer periods of interruption often because of political turmoil and because of changes in state politics, for example, from 755 to 766 and from 780 to 807. In the Song Dynasty (960–1279) the school reopened in 1087, but closed after several short interruptions in 1120 (Volkov 2014, p. 63 and 66). In general, Chinese mathematics began to decline after around 1300.

2.5 Greece and Hellenism

Now entering Greek culture, we are, according to the lore, within the domain where the first prototype and paradigmatic realisation of a genuine textbook has been established, namely Euclid's *Elements of Geometry,* providing at the same type a prototype of Greek deductive mathematical thinking.

Yet, it is not exactly like that. Firstly, Euclid's works are due to the period of Hellenism, after the blossoming of Athenian culture. For this classical Greek period, we do not have preserved manuscripts of a mathematics textbook in use there. Moreover, there is no evidence that Euclid wrote his Elements for teaching aims. For sure, soon after and over millennia, the work has been used for teaching (Fig. 2.11). It became emblematic as a schoolbook, in particular by the selection of its books I to VI and XI to XII, which were used for centuries as representing school mathematics.

ΕΥΚΛΕΙΔΟΥ
ΣΤΟΙΧΕΙΩΝ ΒΙΒΛ. ΙΕ.
ΕΚ ΤΩΝ ΘΕΩΝΟΣ ΣΥΝ-
ΟΥΣΙΩΝ.
Εἰς τὸ αὐτὸ τὸ πρῶτον, ἐξηγημάτων Πρόκλου βιβλ. δ΄.
Adiecta præfatiuncula in qua de disciplinis
Mathematicis nonnihil.

BASILEAE APVD IOAN. HERVAGIVM ANNO
M. D. XXXIII. MENSE SEPTEMBRI.

Fig. 2.11 Cover of the first printed edition of Euclid in Greek, by Simon Grynäus, Basle 1533

In fact, it is exactly for Euclid that the methodological caution applies, as discussed in Chap. 1, regarding the teaching context: “taking it for granted when it should be justified and studied more carefully”. The teaching function had always been taken for granted, but recent research distinguishes between the intentions of the author and its later uses:

> This precaution also eliminates deeply-ingrained confusion between various periods of history or doubtful assimilations, such as the frequent claim that Euclid’s fundamental purpose for the *Elements* was pedagogical. The problem with this assertion is that Euclid and his exact purpose cannot be directly known because no document from the Hellenistic period relates to these questions. What is known for certain is that Euclid’s purpose for his *Elements* may be interpreted as other than purely didactic, and that the first explicit mention of a didactic purpose for the *Elements* only appears some eight centuries later in Proclus’s commentary to its first book (Vitrac 1990, 34-40; Bernard 2010b) (Bernard 2014, p. 41).

Euclid’s *Elements* appears to be a textbook, since it had undergone a process which has been conceived as ‘classicisation’ – of turning to become a *Classic*, as it occurred with numerous texts of Antiquity. And due to this process, it came to be used in teaching: “the process by which these sources were *made* classical can hardly be dissociated from the activities of teaching and learning” (ibid., p. 40). It is highly important to pay attention to this process of classicisation, as the lore had constructed an immediate teaching context for this work, from its composition on. These so often repeated affirmations have to be deconstructed.

In historiography, there is a tendency to understand Euclid’s work as implementing a unity between research and teaching made possible by the two famous institutions of the Mouseion and the Library of Alexandria. This view, which interprets the Hellenistic era in modern terms, however, is a mystification. According to this perspective, Euclid was a scholar who researched and taught at the Mouseion, and who used the Library for his research. Accordingly, his geometry textbook was disseminated by his own teaching and the transmission of his text was ensured by the Library during the Hellenistic and Roman eras.

Such a view, however, is contradicted by known facts about how the *Elements* were disseminated. Their earliest known fragments date from the second half of the third century BCE. And these fragments contain parts with propositions of Book XIII. While these fragments agree in meaning with the text today considered to be authentic, they do not agree in their phrasing (Schreiber 1987, p. 78). The original had probably been written on papyrus, by no means a durable material. The only extant fragment of Euclid on papyrus was found during excavations in Herculaneum and does not literally agree with today’s “authentic” text either. In short, all we know today are the 120 lines of *Elements* written before the fourth century CE and before their text had been established by Theon of Alexandria. Among these 120 lines, only half correspond to the standard version of Heiberg (ibid.). It follows that Euclid’s text must have been communicated in the predominant mode of those times, that is, by oral tradition, and that, therefore, standard version did not exist.

A second argument is that almost nothing is known about Euclid and his biography. The fact that he lived around 300 BCE is deduced from a few hints given by Proklos, and there is no evidence that he worked in Alexandria (see Vitrac 1990, pp. 13 ff.). There is also no conclusive evidence at all that he held a post at the Mouseion.

This leads me to discuss this “Mouseion”. In books on the history of mathematics, it is generally understood that it was something between a university and an academy, bringing teaching and research together under one roof. It should be mentioned that Euclid’s geometry book is generally considered to have emanated from research – quite in contrast to Kuhn’s separation between textbook production and research.

While the structure of the Library of Alexandria is relatively clear and we know not only that it maintained a chief librarian, but we also know the names of numerous scholars who held that post, very little is known about the Mouseion. In its Greek origins, a Mouseion was a centre built to cultivate and worship one of the Muses. Shrines dedicated to them were open porticoes with an altar, but they did not form regular temples. There are historical examples of sanctuaries for the Muses that were primarily the centre of a literary society which met and worshipped them in the shrine and promoted literary competitions.

A particular development of the cult of the Muses was of decisive influence on the founding of the Mouseion in Alexandria: the important role assigned to the Muses in the famous philosophical schools of Plato and Aristotle. Associating the Muses with philosophy was something very popular in the fourth century BCE. Plato’s Academy, apparently, was organised as a body of persons united by the religious purpose of serving the Muses. The personal interest of Aristotle and his successors in the natural sciences has certainly led to a predominance of scientific activities over purely literary activities (see Fraser, vol. I, 1972). The only concrete description of the Mouseion in Alexandria was given by the Roman historian Strabo (63 BCE–23 CE):

> The Mouseion is part of the royal quarter, and hit as a cloister and an arcade and a large house in which is provided the common meal of the men of learning who share the Mouseion. And this community has common funds, and a priest in charge of the Mouseion who was appointed previously by [Ptolemaic] kings, but now by Caesar (quoted in Fraser 1972, p. 315).

Thus, the Mouseion may have constituted a collegiate community of scholars provided with means to undertake studies. It has thus been an early and rare case of a research institution. Systematic teaching, however, had not been intended when it was endowed, and members would probably teach, if ever, privately - and only on their own initiative. As a research institution, it was rare and precocious, as its permanent endowment needed a continuous legitimation for research, a social development not generally accepted until Modern Times. If the Mouseion could indeed boast of such a permanent endowment, this seemed to be due to funds granted by the first Ptolemaic Pharaohs interested in science. Self-administration may have been possible in view of the Mouseion’s religious duty to worship the Muses.

In any case, historical research on the Alexandrian era has shown that not a single one of the great writers, scholars, and scientists known to have been in Alexandria is ever mentioned in the sources as having been a member of the Mouseion, or as having done his studies or taught there. It is much more likely that the impressive scientific achievements of that period were not achieved by the permanent members of the Museum, but rather by scientists relying upon direct sponsorship by the Ptolemaic Pharaohs. In fact, except for the post of chief librarian, held by eminent scientists, the only evidence of positions for scholars is linked to educators of princes – a typical form of patronage that implies at the same time its limitations (see Fraser, vol. I, 1972).

Patronage by princes and kings was the traditional way of promoting science until Modern Times. The fact that eminent scholars depended on this patronage and therefore had to agree with the political intentions of the pharaohs is also clear from the decline of scientific and intellectual life in Alexandria, around the second century BCE, and the exodus of intellectuals and philosophers who fled a political situation that had changed for the worse.

Even the most fruitful first Alexandrian period thus preserved the separation between teaching and researching, fostered by the fact that knowledge was transmitted orally. This persistent structure is demonstrated by one more example, also related to Alexandria. The Roman Emperor Claudius (first century CE) wrote a history of the Etruscans, the "Tyrrhenica", a gigantic work in 20 volumes. Claudius had it deposited in the famous Library of Alexandria. To accommodate the work and store it appropriately required even an enlargement of one of the buildings. It is highly revealing that the Emperor ordered his work to be read publicly once a year. Although "books" existed and a large number of scribes copied them in Alexandria, oral transmission was still necessary to keep the knowledge alive, and to ensure that it was passed on to the next generation.

Therefore, it is a characteristic of both Euclid and his book that the production and dissemination of new knowledge has been kept systematically separate. Although the ancient civilisations were able to teach given knowledge in a relatively stable and tentatively institutionalised form, new knowledge was produced only intermittently with regard to time and geography, and mostly in an isolated manner, outside institutions. Research was funded by private means or by patronage. During the orality period, the knowledge system remained essentially static. Innovation used to be achieved from the outside.

How was Euclid's textbook disseminated (see Murdoch 1971)? In Classical Antiquity, we know that it was studied by several scholars, and that they wrote comments (Proklos). Its transmission after Antiquity is due to the Islamic culture that made it known in Europe. In fact, numerous Islamic scholars have studied Euclid and deepened some of his problems further, particularly in algebra and arithmetic. The role of the Byzantine scholars, however, is often overlooked. Although they do not appear to have had a profound impact on mathematics, they have preserved the texts. Manuscripts from the Byzantine period contain the oldest known versions, and formed the basis for modern critical editions.

Euclid's Στοιχεῖα was essentially conceived as a systematisation of mathematical knowledge developed so far according to its understanding as a deductive science – thus constituting a handbook, not a textbook for teaching. Benno Artmann has shown masterfully how various layers of different Greek conceptual developments have been integrated by Euclid (Artmann 1988). The structure of its 13 "books" evidences the enormous difference and novelty of this presentation of mathematics as compared to the texts discussed so far, from Mesopotamia, Egypt and China – not intending to present procedures for solving problems:

I Geometry of triangles and circles; conversion of surfaces (Pythagoras theorem)
II Geometry of the rectangle and the square
III Circles. their diameters and tangents
IV Regular polygons inscribed and circumscribed in circles
V Proportions theory for geometric quantities
VI Similarity of plane rectilinear figs.
VII Definitions and properties of natural numbers
VIII and IX Properties of potencies and products of natural numbers
X (In) commensurable and (ir) rational straight lines and areas
XI Spatial geometry: solids, spherical surfaces, angles, parallelepipeds
XII Areas and solids: polygons, cones, cylinders
XIII The five platonic solids.

One finds in Euclid's *Elements* an effective restriction to the use of compasses and rulers – not due to foundational principles, as ascribed to Plato, but as a practice. According to studies by Wilbur Knorr, one can remark the conditioned restriction to these instruments in systematisations of knowledge in Euclid's *Elements*. Euclid did not deal with the Delian problem, for example (Schubring and Roque 2016, p. 95 ff.).

Euclid's text has always been presented as inseparable from its diagrams. This conviction has been questioned, however, by Ken Saito's research on the diagrams that have been transmitted with the manuscript versions. He has demonstrated their great differences, not revealing the diagrams in that apparently canonical form as we know them from the current print versions. It was a surprise that these so canonical forms are very recent, stemming from the beginning of the nineteenth century.

One can, therefore, ask what function and form the diagrams had in the first forms of Euclid's *Elements* and in the practice of reading and teaching that followed. Since the culture at that time was largely oral, one can assume that the reader drew the diagrams himself, or a teacher for others ad hoc, for example, in the sand. Indeed, the geometrical problems are formulated in such a way that they enable a respective construction of the explanatory diagram. Among the preserved text fragments of the elements on papyri, fragments were found with figures drawn on the margin. Particularly known are the so-called Oxyrhynchos fragments, excavated in the remains of a Hellenistic city, about 160 km southwest of Cairo. The diagrams visible on the papyrus fragments were evidently drawn by the scribe, without necessarily having a classic model.

Another important Greek mathematical text was the *Konika*, written in the third century BCE by Apollonios of Perga, a city in Asia Minor. The author lived in the Hellenism period and there are hints of some links to Alexandria. His work presents the knowledge about conic sections – the first four of its eight books are regarded as likewise systematisations of earlier works by Greek authors, while the other four books are presenting more Apollonios's own research. As the author had explained himself in his preface, actually a letter to his friend Eudemus: "that the first four books 'belong to a course in the elements,' while the latter four 'are fuller in treatment'" (Fried and Unguru 2001, p. (24)). Michael Fried and Sabetai Unguru, in their careful edition of Apollonios's work, give utmost focus to contextualise it, emphasising that it cannot be called a textbook, at least not in the usual meaning, i.e., for a student, but it is destined to the professional mathematician (ibid., p. [26]). In fact, also later in the Europe of Modern Times, his work was received and used by professional mathematicians – it was not used like Euclid's *Elements* in secondary or higher education teaching as a textbook.

The *Arithmetika* by Diophant can be understood more clearly as a textbook. It was a first work on algebra. It forms a striking break with the dominance of geometry in the classical period. Diophant wrote this textbook in 13 chapters (again: "books"). Practically nothing is known about his life either. There are connections to Alexandria. His lifetime can only be roughly indicated as "fl. 250 CE". Few of his work has been handed down, and in a complicated form. Until recently, only 6 of the 13 books were known in Greek transmission. When the monk Maximos Planudes (approx. 1255–1335), one of the apparently few people interested in mathematics in Byzantium, looked for copies of Diophant's work in his research for mathematical manuscripts in Byzantium, he could only find three copies. All of these manuscripts only contained the six Greek books (Meskens 2010, p. 116, FN 61). One of these copies came from Byzantium to the Marciana library in Venice before 1453 (ibid., p. 132 ff.).

The Arabic translation of the *Arithmetika* was made by Qusṭā ibn Lūqā in Baghdad between 860 and 890; it is known that he also translated the first three books, now only known from the Greek tradition (Meskens 2010, p. 110). One cannot affirm that he translated the entire work:

> It is unlikely, however, that Qusṭā's translation was really complete, that is, that it included all thirteen Books. What we do know is that the first seven Books existed in Arabic translation- Books I~III (and IV) appearing in large part in al-Karajī's *Faḫrī,* and Books IV to VII in our manuscript (Seslano 1982, p. 9).

It was not until 1968 that Fuat Sezgin found a manuscript with four of the missing books in the Astān Quds library in Meshhed (Iran), in an Arabic version, written in 1198, based on Qusta ibn Luqa's translation. Editions of this manuscript were published independently by Roshdi Rashed in Cairo, in 1975, and by Jacques Sesiano, in 1975, as a dissertation, and then in 1982.

According to Sesiano, the ten books that are now available are to be arranged as follows: Greek I, II, III, Arabic I, II, III, IV, Greek IV, V, VI. It remains open where the three missing books should be inserted (Seslano 1982, p. 84). The work contains arithmetic and algebraic problems arranged according to increasing degree of difficulty, by generally formulated tasks and by concrete, numerical solutions. The detailed foreword presents the work as a didactic project: to enable the reader to invent arithmetical problems. The algebraic part is about solving linear and quadratic equations and about problems with right triangles. The small number of surviving manuscripts and the selection of six of the 13 books in the Greek versions suggests that the *Arithmetika* was not an elementary textbook.

Roman culture is distinguished by the development and use of a different type of textbooks, those for land-surveyors – the *agrimensores*. This is the first known type of textbooks for technicians, exposing applications of geometry, which continued to be used in Medieval Times. There has been preserved a variety of these textbooks, known as the *corpus agrimensorum*. Transmitted in a rather confused state, research has progressed to analyse this corpus (Bernard 2014, p. 49).

2.6 India

In the Vedic period of India, about 2500 BCE to 500 BCE, the education of a young Brahman was to memorise, by sophisticated methods, the orally taught Veda, the holy text, composed in form of verses. By the end of this period, the *śulbasūtras* ("Rules of the Chord") were developed as part of larger ritual texts also in the form of verses. One can understand them as a mathematisation of astronomy. Due to their task to construct Vedic ritual altars, their primary aim was not educational.

After a long interruption, mathematical texts written in Sanskrit arose again in the Classical Indian period from about 500 CE to the twelfth century. Remarkably, they pertain to two different strands for applying mathematics: on the one hand, chapters on mathematics within textbooks on astronomy, continuing the strand of "astral science", and on the other hand, texts on "secular mathematics", mostly in the contexts of Jainism. Both text types were still written in verse, as rules, more or less aphoristic, with definitions and procedures (Keller 2014, p. 73).

An important text of the first direction was the *Āryabhaṭīya* of Āryabhaṭa (born 476 CE), of the year 499. His work on astronomy contains one chapter on mathematics, with 33 memorisable verses on elementary geometry, spherical geometry, quadratic equations, arithmetic series, approximation methods for square and cube roots, and tables for sine and sine versus. In particular, the chapter contains a definition of the place-value system.

In Brahmagupta's (598–ca. 665) textbook on astronomy (628), four of the 25 chapters are devoted to mathematics: calculating with quantities, fractions, proportions, and first and second-degree equations – as rules, without proofs. The also astronomical work of Bhāskara I (approx. 600–680) contains a chapter on mathematics, which is a commentary on *Āryabhaṭīya* and contains explanatory examples with verification of the answers.

The texts of the so-called worldly mathematics – or board mathematics because calculations and drawings were carried out on a plate – were found more by chance; for instance the *Bakhshālī* manuscript. Syntheses of the two different forms have been found since the twelfth century. There are two characteristic examples: *Līlāvatī* (on arithmetic) and *Bījagaṇita* ("seed of mathematics", on algebra), both in the form of chapters of the astronomical treatise *Siddāntaśiromāṇi* by Bhāskara II (1114-ca. 1185). In addition to algebra (quadratic and indefinite equations) and spatial geometry, they contain a lot about trigonometry, which is important for astronomy.

2.7 Islamic Cultures

Institutions which one can classify as of higher education level existed in the countries of Islamic civilisation. First, since the eighth century, the *masjid* founded and financed by means of *waqf* the canonical form of a charitable foundation, serving for education in the religious sciences. The so-called foreign sciences were excluded from teaching at the *masjid* (Makdisi 1981, p. 21 ff.). By ‘foreign sciences’ – also called ‘sciences of the ancients’ and suspect because of their ‘pagan’ character (ibid., p. 77) – was understood the Hellenistic knowledge, which the Arab conquerors encountered in Syria and Egypt and apprehended, thus philosophy, mathematics and natural sciences.

The characteristic such institution in classical Islamic civilisation became the *madrasa*. It originated in the twelfth century at the latest, primarily in Syria, Egypt, Iran and Iraq (Brentjes 2014, p. 86). Their main goal was teaching in the so-called religious sciences, in *fiqh* (positive law) and *hadith* (religious tradition) – i.e. for the training of judges (*kadis*) and Islamic legal scholars (*muftis*) – and they also could only be founded by a *waqf*, serving hence a religious objective (Makdisi 1981, p. 35 ff.).

In addition to the religious sciences, or ‘sciences of Islam’, auxiliary sciences were also taught at the *madrasa*. This included Arabic language and grammar. The founder could have chosen to teach the “sciences of the ancients” as well. Even if this was not provided for in the donor’s deed, nothing prevented a student from studying mathematical texts by himself, using the library. A teacher who was equally competent in mathematics could also teach mathematics under the guise that he was dealing with *hadith*. (ibid., p. 80).

Because of this relatively weak level of institutionalised teaching, a different structure was more effective for teaching mathematics. The form of patronage at courts of caliphs, sultans, and emirs has been used extensively. Mathematicians have been employed there for tasks in astronomy and astrology (Brentjes 2014, p. 86); and they could also use their position for teaching in the mathematical sciences.

There is a telling example for this structural pattern of patronage; it serves at the same time to deconstruct another myth. It concerns the *Beit-al-Ḥekma*, House of Wisdom, founded in Baghdad, the new capital of the Islamic Empire. Its foundation is credited to the caliph al-Ma’mūn (reigning 813–833), as a library and for the translation of foreign books.

The *Beit-al-Ḥekma* has been anachronistically transfigured as a university; in order to rule out such wrong attributions, recent research calls the institution a translation “bureau” (Gutas 1998, p. 56). In fact, *Beit-al-Ḥekma* is the translation of the term in Persian for ‘library’; only translations of works from Persian can be identified there (Gutas 1998, p. 58). The movement of the translation of the works by Greek philosophers and scientists was independent of the *Beit-al- Ḥekma* but also initiated and realised by the Abbasid caliphs - and began intensively even earlier, by the caliph al-Manşūr (754–775), the founder of Baghdad. It was indeed al-Manşūr who had asked the Byzantine emperor to send him Euclid's *Elements* and who made it translate from Greek into Arabic (ibid., p. 32) – a first translation that was later improved. At least one source confirms that al-Khwārizmī was employed at *Beit-al-Ḥekma* "in the service of al-Ma'mūn" (ibid., p. 58). Since there are no known translations made by al-Khwārizmī, and since he dedicated his algebra book to this caliph, one can assume he was officially employed as a librarian there.

A great number of textbooks have been developed and used for teaching mathematics in Islamic countries. Among the "foreign" implemented ones, Euclid's *Elements* occupy a privileged place – the Islamic world was very interested in it. Caliph al-Ma'mūn sent another diplomatic mission to Byzantium, asking again for a copy (de Young 1984, p. 148). The first copy obtained was translated by al-Ḥajjāj ibn Yūsuf b. Maṭar (fl. around 800), for the caliph Hārūn al-Rashīd (r. 786–809). He made a second translation, probably using the second copy, for the caliph al-Ma'mūn; it is called the *al-maʾmūnī* translation:

> The anonymous preface to a sixth/twelfth-century copy of Abū l-Faḍl al-Nayrīzī's (died ca. 309/922) commentary characterizes it as an edition of the previous translation which was corrected and shortened as well as modified by changing its language and filling its gaps to satisfy the interests of al-Maʾmūn's courtiers (quoted from de Young and Brentjes 2013, p. 2).

A new translation of the Elements was made by Isḥāq ibn Ḥunayn (died 911), around 879; it is transmitted in an edition by Thābit ibn Qurra (died 901); see Fig. 2.12. These main versions were widely used in the Islamic world, in particular, not only copied but also commented on and extended. The translation by al-Ḥajjāj became the basis for ibn-Sīnā's (adaptation of Euclid's *Elements*, which he had elaborated as a part of his Encyclopedia, *Kitāb al-Shifā*.[3] In the thirteenth century, for instance, there were at least six editions of the Arabic Euclid.

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[3] Ibn-Sīnā's encyclopedia had four pars on mathematics: arithmetic, geometry, mathematical astronomy and music (de Young 2018, p. 78).

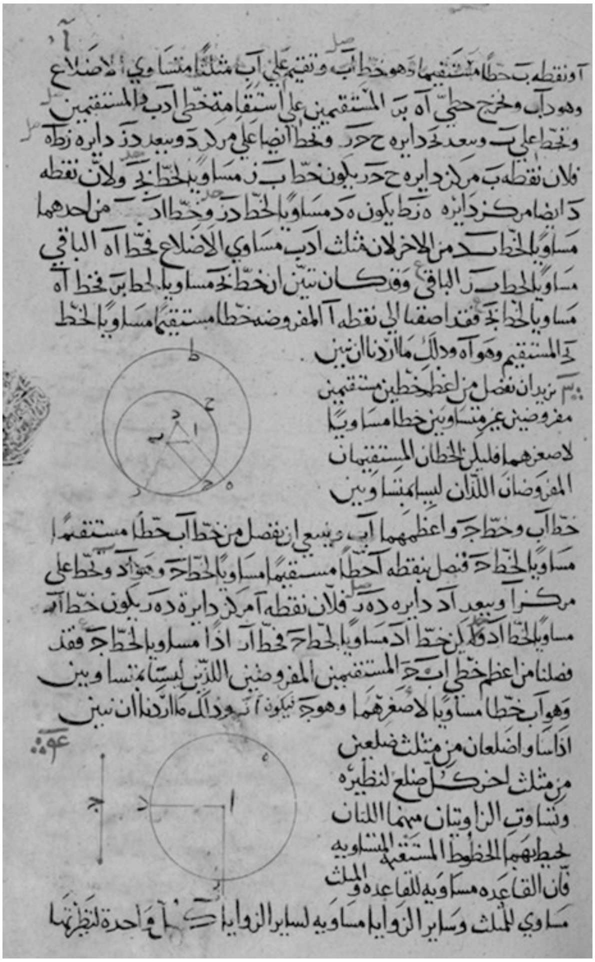


Fig. 2.12 Propositions 2 and 3 of Euclid Book I, from the manuscript of the translation Isḥāq ibn Ḥunayn, edited by Thābit ibn Qurra, fol. 13. Courtesy by the Bodleian Library, Oxford

The edition by Nasīr al-Dīn al-Tūsī (1201–1274) became a most influential one. He inserted so many comments and additions that the volume doubled in length. “It became the standard version studied in *madrasa* classes or with private tutors in many cities across the Islamicate world, replacing the third/ninth-century translations” (de Young and Brentjes 2013, p. 3). It was used for teaching even during the nineteenth century, in the Moslem part of India. Its Book I was printed as a lithograph in 1873, due to a curricular reform of the Indian madrasa, by Muḥammad Barakat (de Young 2012, p. 134; see Fig. 2.13).

Fig. 2.13 Propositions 6 and 7 of Book I in Barakat's adaptation of at-Tūsī's Euclid. (Barakat 1873)

The first complete translation of al-Tūsī's *Elements* into Persian language was produced by Quṭb al-Dīn al-Shīrāzī (1236–1311) (Gregg 2007). Moreover, Quṭb al-Dīn al-Shīrāzī elaborated an Appendix to this translation, which is highly remarkable due to a diagram by which he intended to integrate all the diagrams necessary for Book I of *Elements* (de Young 2013). I have analysed how he had succeeded in uniting the diagrams for the 48 propositions of Euclid's Book I in just one diagram (Fig. 2.14). It proved that it worked reasonably well. And the diagram shows the author's intention to generalise the Euclidean propositions, which are characterised by their synthetic nature – hence to treat each proposition as an independent one, avoiding to stress common properties. Al-Shīrāzī, however, aimed at showing conceptual relations between different propositions, representing them by the same diagram element. Figure 2.14 shows that he referred to propositions 27, 28, and 29 of Book I with the same elements of his "combined diagram".

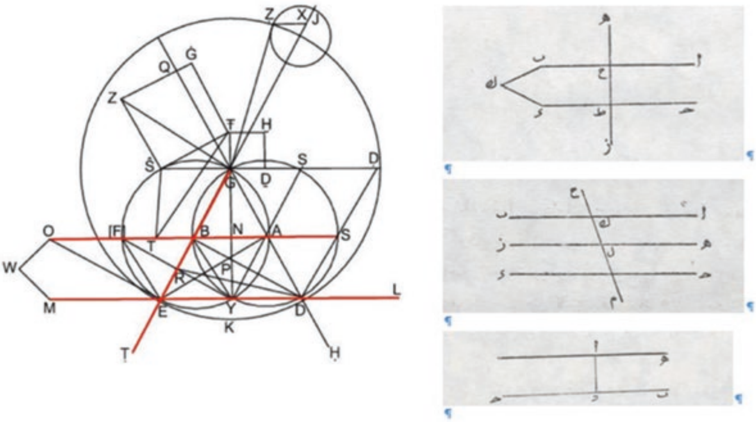


Fig. 2.14 Unified diagram for I, 27 to 29 by al-Shīrāzī (his Arab letters substituted with Latin ones) (Schubring 2018, p. 271) and the corresponding separate diagrams by ibn-Sīnā (Sabra 1977, pp. 50–52). Note that diagrams for I. 27 and 29 of ibn-Sīnā are "over-specified": one asks there alternate angles, and not rectangular ones

The diagrams of 33 of the 48 propositions were in fact united in that single diagram. Fifteen others, quite easy to draw, were left for "free" drawing. One can resume:

> Al-Shirazi's comprehensive figure expressed his intention to focus more on common traits, to emphasise related properties, not to be fixed on particularities. Thus, one can say that this approach of al-Shīrāzī signifies a step towards generalisation, of overcoming the synthetic conception of Greek geometry where each case was studied separately. Therefore, it not only resulted in a reduction of repetitiveness but meant also a step towards the analytical method aiming at the inter-connections between the propositions and the generality of the statements (Schubring 2018, p. 273).

Besides the reproduction of translations of Euclid's *Elements*, numerous textbooks on other mathematical subjects have been produced in Islamic civilisation. But particularly characteristic for this culture is that those who taught used to produce commentaries on the books they were using in teaching. This practice has been assessed even as having caused a "decay of mathematics": "an ominous and regrettable consequence of this succession of abridged works, commentaries on them, commentaries on commentaries, epitomes of such commentaries, and commentaries on such epitomes" (Abdeljaouad 2012, p. 8).

The great number of textbooks produced in the Islamic civilisation has surely been enhanced by the introduction of paper and its use for writing mathematical manuscripts. The introduction of paper, probably transmitted from China, is attested first for Persia around 700 CE, where Samarkand became the centre of paper production. In Baghdad, the production of paper was initiated in 795.

Algebra, by al-Khwārizmī (780–850), is surely the best-known textbook of Islamic mathematics. It had enormous influence and importance by establishing algebra as a proper subdiscipline of mathematics. It was written entirely in rhetoric style, without symbols. Already its title *al-kitāb al-muhtasar fi hisab al-jabr wa-l-muqābala* indicates the basic procedures for normalising equations: removing subtractive terms and removing multiple appearances of a variable in the same power. Normalising right and left sides of equations, he established six standard types of linear and quadratic equations, based on coefficients being positive numbers (Hogendijk 1992, p. 73 f.).

Abdeljaouad distinguished two types of books on arithmetic: one theoretical, regarding the "science of numbers", and the other one practical, regarding "the art of calculating and its applications in daily life" (Abdeljaouad 2005, p. 7). Tellingly enough, he listed only three books for the theoretical treatment, and, likewise telling, they are all parts of encyclopaedias: the chapter arithmetic in the *Rasā'il Ikhwān as-Ṣafā* of the ninth century, again a chapter of the encyclopedia *Rasā'il Ikhwān as-Ṣafā a* (about second half of tenth century) and the chapter *al-arithmatīqa* in ibn-Sīnā's encyclopedia *Kitāb al-shifā,* after his chapter on geometry (see above).

Very numerous are the books on practical arithmetic, also due to the importance of calculating inheritances. For better analysing this great number, Abdeljaouad classified them in three groups:

- The first type is called finger arithmetic; their style is rhetoric – all is expressed in words, without any symbol. The user of such a textbook has to apply various mnemotechnic methods and rely on well-defined positions of his fingers for effecting the calculations. They served mainly for dealing with daily life problems. A good example for this type is the book by Abū l-Wafā (died 998), very often copied and commented on, and it was used until the seventeenth century: *Kitāb ma yaḥtāju ilayhi al-kuttāb wa l-ᶜummāl min ᶜilm l-ḥisāb* [Book on Arithmetic, necessary for scribes and merchants].
- The second type are textbooks of sexagesimal arithmetic. These textbooks, based on the sexagesimal number system, were elaborated, on the one hand, for solving daily life problems, and, on the other hand, for all types of astronomical calculations. A typical title of this group is that by Ibn al-Majdī (died 1447): *Kashf al-haqâ'iq fî hisâb ad-daraj wa l-daqâ'iq* [The unveiling of the truths about calculating the degrees and the minutes]. Sexagesimal arithmetic remained very long in use, until the nineteenth century.
- The third type were the textbooks of Indian arithmetic. The introduction of the Indo-Arab numbers occurred by 800, and is represented by two important textbooks: the first being by al-Khwārizmī and the other one by al-Uqlīdisī (fl. 950). The book by Al- Khwārizmī is his second important textbook. Strangely enough, no original version is known so far; hitherto, its text was known only by various excerpts translated into Latin. Menso Folkerts, however, succeeded in finding a complete Latin version and published an edition of it (Folkerts 1997). Right at the beginning, the aim of the book is explained:
- We decided to explain how the Indians count using nine symbols and to show how, thanks to their simplicity and conciseness, these characters can express all numbers. We will thus facilitate the task of those who want to learn arithmetic,

> that is to say, both large numbers and small numbers and all that relates to it: multiplication, division, ... (quoted from Abdeljaouad 2005, pp. 32 ff.)

Remarkably, al-Khwārizmī admitted only 1–9 as numbers. Zero, represented by a small circle, was not recognised as a number. His book has the following structure:

- Introduction of numbers and the values of their positions.
- (1)–(5) Integers: addition and subtraction, duplication and division into two halves, multiplication and division, fractions in the decimal and sexagesimal systems.
- (7)–(9) Multiplication and division of sexagesimal fractions, addition, subtraction, duplication and division into two halves
- (10)–(11) Multiplication and division of common fractions.
- (12) Extraction of square roots.

The book *Kitāb al-fusūl fi-l ḥisāb al-hindī* [Book of the sections of Indian calculation], written by al-Uqlīdisī around 952, extended the book by al-Khwārizmī, including all arithmetic known then – emphasising its Byzantine and Arab origin. Only recently, a manuscript of this book was discovered. It was edited by Saydan (Saidan 1978), who also restricts the Indian numbers to nine symbols.

The Indian numbers were then generally called *al-ghubār* numbers, sand numbers, and this arithmetic was called *hisāb al-ghubār*, hence the English title The Sand Reckoner. The use of Indian numbers was for the first time tied to the use of a *takht*, a board covered with sand on which operations were carried out with a stylus. Yet, this mode met certain resistance; therefore, al-Uqlīdisī recommended, an alternative mode:

> [the Indian numbers] require the use of *takht*, but we say it is an art that requires the use of a tool [...] that is easy to use and inexpensive. If some do not like it because of the sand which dirties the hands and can hurt some fingers, we say that we can use a stylus to write on one end and to erase on the other end [. . .]. But we propose to all the use of paper (Saidan 1978, p. 40).

It is highly remarkable that here the use of paper, ink, and pen is proposed! Al-Uqlīdisī devoted one quarter of his book to explain this quite new technique.

Another extraordinary work is the book *Miftāḥ al-Ḥisāb* [Key to Arithmetic], by Jamshīd Ghiyāth ad-Dīn al-Kāshī (1380–1429). He was a mathematician and astronomer working at the observatory in Maragha (today, Iran), invited by Ulugh Beg. Its 108 pages, in the 1977 Arabic edition (Fig. 2.15), and 220 pages in the bilingual English-Arabic translation (Aydin and Hammoudi 2019), is actually a handbook rather than a textbook. Yet, the term ‘key’ in its title has been used later in Europe in mathematics textbooks. And it is noteworthy for its first elaborate establishment of decimal fractions.

Fig. 2.15 Cover of Al-Kāshī, *Miftāḥ al-Ḥisāb*, Damascus edition of 1977 by Nabulsi

In his second book, he announced to have found particularly interesting fractions whose denominators are potencies of 10. He called them *al-kusḥr al-‘a^{c}shariya* [decimal fractions]. Instead of writing them vertically as Indian and Islamic predecessors:

358
501
1000

he wrote them in this form:

a tenth ($\frac{1}{10}$), a tenth of second order ($\frac{1}{100}$), a tenth de third order ($\frac{1}{1000}$), etc. (see al-Kāshī 2019, p. 223).

Al-Kāshī commented:

> So, this is as if we divide one into ten parts, and divide each tenth into ten parts, then each part from them into ten parts, and so on, until the end. We call the first parts tenths, because they are,[4] the second parts seconds of tenths, the third parts thirds of tenths, and so forth,

[4] One or more words are missing in this translation.

until the end so that the orders of the fraction and the integers are on the same ratio analogously to astronomers' arithmetic. We call them the decimal fractions. We must write the tenths on the right of the units, the second tenths on the right of tenths, the third tenths on the right of its second, and so forth until the end. The integers and the fractions will be on the same row (al-Kāshī 2019, p. 219).

Al-Kāshī's textbook represents one of the last highlights of the already exhausting productivity in the Eastern Islamic regions. Productivity continued, however, in the Western regions: in Al-Andalus, until its transformation into Christian Spain, and in particular in the Maghreb (see Lamrabet 2020).

2.8 European Middle Ages

After the long centuries of the Dark Ages, revival of learning and science began in Western Europe, first with initiatives by the Carolingian king and emperor Charlemagne, around 800, then developing more strongly from the eleventh century, due to emerging urban culture. These developments reveal the same twofold pattern as already remarked in the Islamic civilisation: on the one hand, a theory-based pattern of teaching; on the other hand, a practice-oriented pattern.

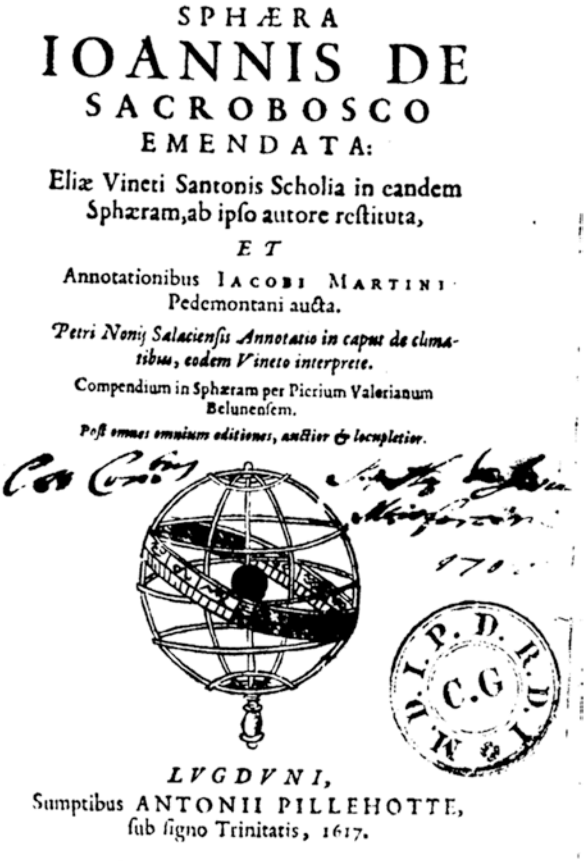
The theory-based teaching was represented by lectures at universities, which had been emerging in Western Europe since about the twelfth century. And this teaching was characterised by the dominance of the textbook. Due to their existence only as manuscripts, the practice of teaching was determined by orality – the lecturer had, as already the term indicates, to read aloud from the textbook. And the lecturer had to watch that the students would have noted correctly what he had dictated. The role of the teacher was hence to serve the handbook, and the student had to passively receive it (Fig. 2.16).

Fig. 2.16 Lecturing in Medieval universities

The form of teaching at these universities was rather analogous to that at the *madrasa*. Makdisi has argued that these served even as models for the universities in Western Europe (Makdisi 1990). The marginal role of mathematics in the Medieval universities, also somewhat analogous to the *madrasa*, was essentially due to the conception of the *septem artes liberales* and to the functioning of the Arts Faculty within the universities.

The *trivium* constituted the major part of the curriculum in the Arts Faculties of the universities of the Paris model (*artes* faculty as a preparatory course before continuing in one of the three higher faculties), while the *quadrivium* was only a minor part of the curriculum. In addition, there were no specialised lecturers at this faculty. Lectures were given by the *baccalaurei*, who continued studying at one of the higher faculties after completing their *artes* studies. The texts to be read were distributed by lot to the *baccalaurei* before the beginning of each semester (Schöner 1994, p. 62). Consequently, they needed not to be specialists of the texts they were reading to the students.

The lectures of the quadrivium were quite elementary, based on the first books by Euclid, the *Tractatus de sphaera* (Fig. 2.17) – a text on popular



SPHÆRA
IOANNIS DE
SACROBOSCO
EMENDATA:
Eliæ Vineti Santonis Scholia in eandem
Sphæram, ab ipso autore restituta,
ET
Annotationibus IACOBI MARTINI
Pedemontani aucta.
Petri Nonij Salaciensis Annotatio in caput de climatibus, eodem Vineto interprete.
Compendium in Sphæram per Pierium Valerianum
Belunensem.
Post omnes omnium editiones, auctior & locupletior.

LVGDVNI,
Sumptibus ANTONII PILLEHOTTE,
sub signo Trinitatis, 1617.

Fig. 2.17 Cover of an edition of Sacrobosco's *Sphaera* textbook. (Lyon 1617) astronomy by the British monk John Sacrobosco (c. 1195–1256) – and *Computus*, an application of astronomy for future clergy to be able to calculate the dates of ecclesiastical holidays. In the rudimentary system of cathedral schools, established due to Charlemagne's initiatives, books used for teaching mathematics were versions of Boethius's (died 524) Arithmetic and of a Geometry ascribed to him (Høyrup 2014, p. 111 f.). Thanks to the translation movement, from Arabic to Latin, in the late tenth century, led by eminent translators Gerbert d'Aurillac (946–1003), Adelard of Bath (1080–1152), and Gerard of Cremona (1114–1187), Latin translations of Euclid became available and were used in university teaching. Lectures on Euclid might have been restricted to his first four books (Høyrup 2014, p. 117).

The Fibonacci Tale

Another practice-oriented pattern arose from the establishment of ever more strongly elaborated and disseminated commercial arithmetic, as a characteristic of emerging forms of capitalism in urban centres, most markedly in Italy, and as a transmission from the Islamic arithmetic. A group of professionals, the *maestri d'abbaco*, practised this commercial arithmetic, wrote textbooks about these practices known as *liber d'abbaco*, and founded schools for training in these practices: the *scuole d'abbaco*.

Traditionally, this transmission is attributed to the work of Leonardo Fibonacci (ca. 1170–1240), who had learned Islamic mathematics in his youth in the Maghreb – when his father worked there as commercial representative for the Tuscan city of Pisa. Frank Swetz has even claimed that his famous textbook *Liber Abbaci* (1202) had served as model for arithmetic textbooks for over 400 years (Swetz 2007, p. 1).[5]

However, this claim does not only neglect the differences between the arithmetic textbooks as arising in the various European regions, from the end of the fifteenth century, but attributes also the transmission of Islamic mathematics to Europe to only one single transmitter and in just one country. In fact, Høyrup called it "conventional wisdom" that the Italian abacus mathematics and its later appropriation in Catalonia, Provence, Germany etc. should have been only transmitted and triggered by this "narrow and unique gate", and stated in contrast:

> However, much evidence – presented both in his [Fibonacci's] own book, in later Italian abbacus books and in similar writings from the Iberian and the Provencal regions – shows that the *Liber abbaci* did not play a central role in the later adoption. Romance abbacus culture came about in a broad process of interaction with Arabic non-scholarly traditions, at least until ca. 1350 within an open space, apparently concentrated around the Iberian region (Høyrup 2014, p. 219).

Proof that Fibonacci acquired his knowledge not only in the Maghreb, but also on further travels, especially in Spain, is evidenced by his introduction of the *liber abaci*:

[5] Frank Swetz in his review of the English translation by Laurence Siegler: "In content and format *Liber abaci* established a genre for European commercial arithmetic books for the next four centuries".

> After my father's appointment by his homeland [the city of Pisa] as state official in the customs house of Bugia for the Pisan merchants who thronged to it, he took charge; and, in view of its future usefulness and convenience, had me in my boyhood come to him and there wanted me to devote myself to and be instructed in the study of calculation for some days. There, following my introduction, as a consequence of marvellous instruction in the art, to the nine digits of the Hindus, the knowledge of the art very much appealed to me before all others, and for it I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth and learned the give-and-take of disputation (quoted from Høyrup 2014a, S. 220).

Høyrup, who has extensively researched about the emergence of arithmetic textbooks in Medieval Western Europe, has recently synthesised this research in a book where he rectifies the lore of Fibonacci's book as the only source. "Nine complete or fairly complete manuscripts survived"; they all derive from Fibonacci's revised version of 1228 (Høyrup 2022, p. 57). A critical edition was published by Enrico Giusti (2020). About Fibonacci's revision, Høyrup asserts: "Fibonacci conserved a master copy of the 1202 version, and inserted new material into it while removing what had become redundant or what he did not like at second thoughts […]. All manuscripts were made from this evolving master copy" (ibid., p. 58).

Fibonacci uses the Hindu-Arab numerals, as all the *abbacus* texts are doing. But Høyrup warns that their introduction in Western Europe is not due to him; there were earlier occurrences of it already in Spanish contexts, for example, in the twelfth century Latin translation of al-Khwārizmī (ibid., p. 55). As Høyrup emphasises, his *Liber abbaci* differs from the later tradition of *abbacus* textbooks: Fibonacci's aim was to "apply a theoretical perspective on practical arithmetic". This is evidenced by two different manners of characterising the applied procedures: either by "secundum vulgi modus" (in the vernacular way) or "secundum artem" ("magistraliter" or according to the art) (ibid., p. 61). Høyrup provides several examples for the two manners. Moreover, his book is written in Latin, while all the later commercial arithmetic books are written in vernacular.

The first chapters expose the basic arithmetic operations with integers. The division in chapters already shows the treatment given to fractions and continued fractions. Operations with "mixed numbers" follow, i.e., operating with integers and fractions. Thereafter, the chapters deal with the rule of three (not named as such) and its various applications in commerce, in alloying. Chapter 12 deals with problems addressing issues of algebra: summation of arithmetic series or of ascending squares, "proportions of numbers" as ratios; first-degree problems are solved by means of the single false position. Some second-degree problems are solved with geometric arguments. There are even recreational problem types ("finding a purse", "buying a horse") (ibid., pp. 65 ff.). Høyrup emphasises the explanation of perfect numbers as another bit of theoretical mathematics (ibid., p. 94). Then follows the part in which the Fibonacci numbers are taught from the engendering of ever new pairs of rabbits (ibid.). A series of indeterminate problems in chap. 13 is characterised as "algebraic procedures" (ibid., p. 97). Chapter 14 deals with square and cube roots, exposing monomials and binomials evidencing thus more algebraic issues, and the difference to the abacus textbook tradition, which according to the lore follow Fibonacci as their model. Chapter 15 includes problems in geometry that are related to Euclid's Elements.

Høyrup analyses various later *abbacus* texts whether they confirm the traditional lore. The first is the *Tractatus algorismi*, written in 1307 by Jacopo da Firenze, of unknown biography – who despite his name was then living in Montpellier, in Southern France (Provence). The text is written in Tuscan. It shows influences of Sacrobosco's *Algorismus vulgaris* (about 1225). After the introduction of the Hindu-Arab numerals and the operations of pure arithmetic, finishing with fractions, the rule of three is taught, for basic commercial techniques. These sections are followed by 39 mixed problems, basically commercial ones, and some of recreational nature. Other problems teach to transform prices into other monetary systems. The book follows with a section on practical geometry, measuring areas. The next section deals with coins, teaching how to exchange for trade with other regions. Alloying practices are the last subject. Algebra is absent in this book (Høyrup 2022, pp. 11 ff.).

A considerable number of more *abbacus* textbooks are analysed while comparing to Fibonacci's text. Prominent examples are *Livero de l'abbecho*, written in Umbria not much later than 1310, and the *Libro di ragioni*, from Pisa. Although at first sight looking like a confirmation of the lore, they "take next to nothing from the *Liber abbaci*" (ibid., p. 157). He even states: "On the basic level we find everything that was taught in the abbacus school; nothing on this level comes from Fibonacci" (ibid.).

The *Libro di ragioni* (book of problems), written by the early fourteenth century, discusses prime numbers and continued fractions, but "there is no reason to conclude from this eclectic treatment of fractions that the present *Libro di ragioni* was inspired by the *Liber abbaci*, and even less to find a deliberate attempt to emulate Fibonacci's ways". Rather, various of his practices indicate that "the present writer had direct contact to the Maghreb" (ibid., p. 163).

Since the thirteenth century, the *scuole d'abbaco*, schools for training in techniques of commercial arithmetic, spread from Florence, the emerging commercial centre of Western Europe. The *maestri d'abbaco*, as heads of these schools, wrote numerous writings on these techniques. The production of these textbooks has been extensively researched (see van Egmond 1976). Van Egmond published in 1980 a catalogue of the Italian arithmetic textbooks, which proved until today to be quite complete (Høyrup 2022). In view of the importance of these schools for the flourishing of their economy, several city governments, which now appeared as new organs for the organisation of teaching, at least monitored the functioning of these schools and thus gave them a certain "public" character. Recent research confirms that a significant proportion of youth growing up in urban centres – around a third of them – attended the 2-year *scuole d'abbaco* and learned practical arithmetic:

> From around 1260 onwards, such schools were created in the commercial towns between Genova, Milan and Venice to the north and Umbria to the south. It was attended in particular by merchants' and artisans' sons, but patricians like Machiavelli and even Medici sons also visited it (Høyrup 2014, p. 120).

From the end of the fifteenth century, the development of commercial arithmetic in Western Europe eventually led to the emergence of algebra – actually in a plurality of forms (see Rommevaux and Spieser 2012).

Besides the development of arithmetic within the needs for commercial arithmetic in urban civilisation, the most intense use of geometrical knowledge in Western Medieval societies was for the construction of the Gothic cathedrals. Different from the Romanic churches, which was the former style, Gothic cathedrals afforded more extended geometrical knowledge and constructing techniques, due to their markedly higher structures and the systematic use of vaults. Our understanding about the kinds of geometric knowledge available to the Medieval mason masters is very restricted, since all their professional knowledge was for centuries transmitted only orally and kept secret:

master-masons did have to promise that they would not reveal the secrets of their trade or craft to anyone outside it. The statutes of the Paris masons demanded it already in 1258 (Rykwert 1987, p. 17).

Shelby describes the training of the stone-masons or master masons as occurring within the craft as

> education into the traditions of their craft, whereby the technical knowledge required in design and construction was transmitted from father to son, from master to apprentice, from learned journeyman to those who were less learned in the craft traditions. Since the geometry of the masons was an essential part of that technical knowledge, mediaeval master masons would normally have acquired their geometrical knowledge in the same way that they acquired the rest of their knowledge and skill in building - by mastering the traditions of the craft (Shelby 1972, p. 398).

This craft was organised like all the other professions but in a different manner. It did not hierarchically separate people as apprentices, masters, etc., it rather united all levels in one guild – in the German case, the *Steinmetzbruderschaft*, the stonemasons fraternity. *Bauhütte* was the German name for the kind of barrack, or building, located directly at the side of the building to be constructed, serving as a worker's hut for the stonemasons and other workmen during the construction of the cathedral. (Binding 1980). In Italy it was called *Opera del Duomo*. Only one manuscript is a source about the geometrical knowledge of the stonemasons at the Medieval *Bauhütten.* It is called *Sketchbook* and was written around 1235 by the French Villard de Honneycourt, showing drawings of constructions collected during travels in France, Switzerland, and Hungary. And there are several booklets about German *Bauhütten*, the so-called *Baumeisterbücher* or *Werkmeisterbücher*, published right at the beginning of the printing press, from 1483, by Mathes Roriczer, himself a *Dombaumeister*. The techniques described and the drawings focus mainly on two aspects: the construction of nested vaults and of pinnacles:

> Pinnacles (also called violas) are small decorative towers to crown buttresses and flanking *Wimpergen*, the Gothic ornamental gables (*Wimperg*, originally wintperge = sheltering from the wind: protective gable). The pinnacle consists of a shaft or body and a helmet or *Risen* (also giants) whereby the shaft can also be designed as a tabernacle – a cavity to accommodate figures (Scriba and Schreiber 2010, p. 233; my transl.).

Constructing nested vaults starts from "Vierung über Ort" (crossing over place):

> In a square that serves as a ground plan, a second square is set by connecting the middle of the sides, after which this process is repeated several times. By rotating every second square around the centre by 45, a series of nested squares with parallel sides is created […]. Two consecutive squares have as ratio of their sides 1: $\sqrt{2}/2$; they form the higher-lying cross-sections of the tapering pinnacle turrets, which are to be transferred in sequence from the foot at precisely defined intervals. After the body of the pinnacle was designed in this way, Roriczer describes the construction of the *Risen* (helmet, pointed roof) and then the design of the decorations (flowers and foliage) […]. Like a recipe, the stonemason is instructed to pick up certain straight lines with the compass and transfer them to the workpiece (ibid., p. 237).

Shelby has edited and translated two of these *Baumeisterbücher* (Shelby 1977). Figure 2.18 gives an example of construction drawings for a cathedral Bauhütte.

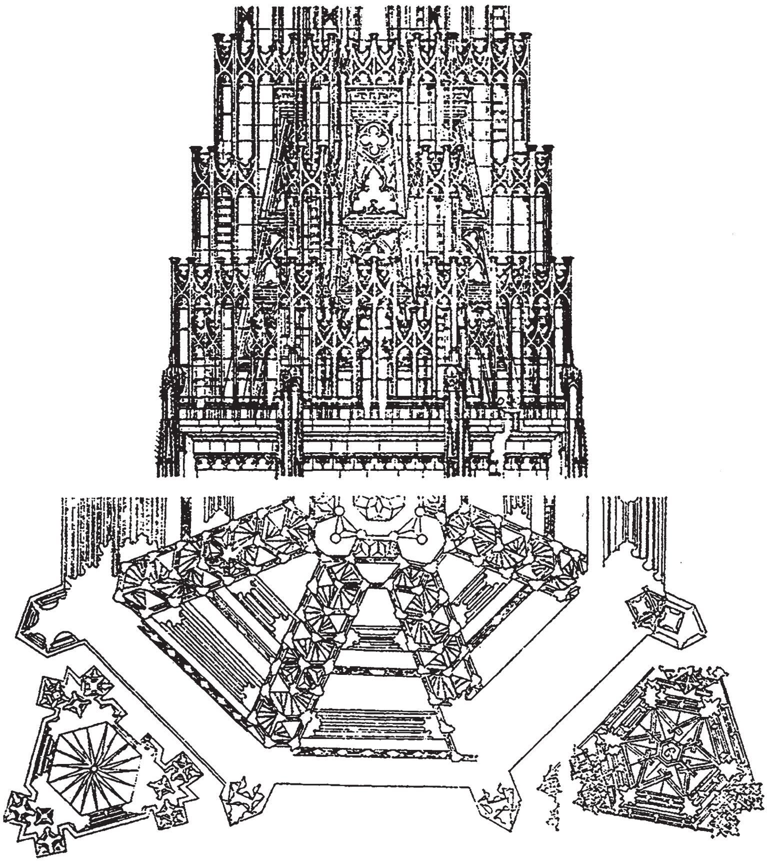


Fig. 2.18 Strasbourg Cathedral. Crack drawing by the Werkmeister Johann Hültz, before 1439 (Binding 2013, p. 203)

Chapter 3
Textbooks in the Era of the Printing Press: The Emergence of Modern Textbooks

3.1 Conflicts in Introducing the Printing Press

The printing press was invented by Johannes Gutenberg (1400–1468) in 1445. The main innovation consisted in the invention of movable letters, combined with the invention of a hand-held casting instrument for the production of these letters, with an alloy of tin, lead, and antimony that he also established, as well as an oil-based printing ink. These inventions were complemented with the development of the printing press; the integration of all these elements enabled the rapid reproduction and manufacture-like printing of books. From 1450 Gutenberg printed on the one hand dictionaries, short grammars, letters of indulgence and calendars and on the other hand the Gutenberg Bible (1452–1454).

A large number of studies have been devoted to this invention and the process of its dissemination; a very instructive example among such books is Elizabeth Eisenstein's volumes *The Printing Revolution* (1983). Throughout this literature, the story of the invention of the printing press and its aftermath is told as the story of a landslide victory. The obstacles and resistance to this triumphant career have been denied and neglected, a fact that is very evident to me from my own studies as to how long it took for the first printing house using Arabic letters to be installed in the Ottoman Empire. Although in Istanbul there had been presses in the hands of foreigners and non-Muslim minority groups since 1493, the first Arab press was not authorised until 1726: the delay was due basically to strong resistance by the numerous group of the copyists and by the muftis being suspicious of a possibly uncontrolled dissemination of publications; in fact, books of the religious sciences were excluded from printing, and censors had to verify the conformity of the printed text with the original (cf. Schubring 2001, pp. 65–67).

The bibliography of the most important and classic works on the history of the press does not contain any category reserved for resistance or rejection (Corsten and Fuchs 1988/1993). I have been able to find just one single study that mentions the problem of delays in press releases (Giesecke 1991). This voluminous study based

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_3

on modern concepts in communication theory and systems theory contains a brief chapter on the contemporary critical discussion of innovation. The criticism reported by Giesecke is very similar to the one that provoked the resistance against the press in the Ottoman Empire.

The literature on the history of some European universities provides indications of at least some resistance against the press: the new technology was seen as a threat to the traditional system of reading texts aloud, to orality. While attempts had been made during the Middle Ages to reform the passive role of students and move teaching beyond just reading texts aloud, this time-honoured and effort-saving practice had remained unchanged. A structural effect was that the printing press tended to blur the traditional separation between teaching and research.

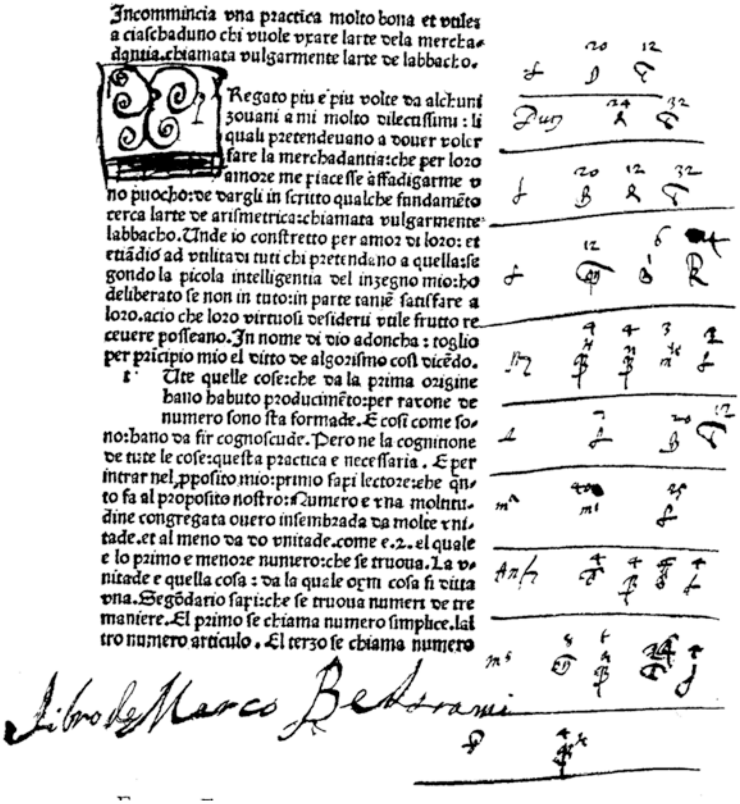
Indeed, the impact of the press worked, combined with humanism, to force universities to carry out internal reforms, to reformulate their curricula and teaching practices, to introduce new disciplines (history, Greek, philology), and to elevate the status of marginal disciplines, as was the case with Mathematics.

After all, universities were successful in integrating the new technologies: many of them hired their own printers for their publications, and these became members of the university corporation. The ease of reproducing texts exempted *magistri* and *regentes* from verifying the fidelity of the transcriptions of dictated texts made by their students.

For the most part, these reforms did not emanate from within universities. Territorial sovereigns decreed innovations, or the universities themselves responded to external pressures for change. Eventually, the integration of printed matter into university teaching practice weakened the traditional role of oral transmission: learning was no longer restricted to mere passive listening, as students now had opportunities to become active and do some studies on their own. Teaching opened up to new ideas; the knowledge system was no longer static. This was also evidenced by mathematics textbooks. Hence, the traditional functioning of the textbook triangle (see Sect. 2.1.) began to change: in principle, the teacher could from then on assume the leading role. There emerged even first textbooks in the degenerate manner of functioning – for self-learning (see below).

3.2 The First Printed Mathematics Books: For Commercial Context

The first printed textbooks were intended for commerce and business. This fact aligns with the framework for mathematics in Islamic civilisation reported in the previous chapter: practical arithmetic was the best developed and most widely accepted form of mathematics, a fact already true for the likely very first printed mathematical text, the so-called *Arithmetica di Treviso*: *Larte de Labbaco*, a commercial arithmetic book of 1478 written in Venetian Italian (Fig. 3.1). Subsequent printed texts appeared in the following decade. The second printed mathematics book was published in Spain, in the kingdom of Aragon, in 1482: *Suma de la art de arismetica*, by Francesc de SantCliment (Salavert 1990, p. 67). It was followed by Ulrich Wagner's *Bamberger*


Incommincia vna practica molto bona et vtiles a ciascbaduno chi vuole vxare larte dela merchadantia. chiamata vulgarmente larte de labbacho.

Regato piu e piu volte da alchuni zouani a mi molto dilectissimi: li quali pretendeuano a douer volere fare la merchadantia: che per loro amore me piacesse affadigarme vno puocho: de dargli in scritto qualche fondamento cerca larte de arismetrica: chiamata vulgarmente labbacho. Unde io constretto per amor di loro: et etiandio ad vtilita di tuti chi pretendano a quella: segondo la picola intelligentia del inzegno mio: ho deliberato se non in tuto: in parte tamen satisfare a loro. acio che loro virtuosi desiderii vtile frutto receuere posseano. In nome di dio adoncha: toglio per pricipio mio el ditto de algorismo cosi dicendo.

Tute quelle cose: che da la prima origine hano habuto producimento: per rasone de numero sono sta formade. E cosi come sono: hano da fir cognoscude. Pero ne la cognitione de tute le cose: questa practica e necessaria. E per intrar nel proposito mio: primo sapi lectore: che quanto fa al proposito nostro: Numero e vna moltitudine congregata ouero insembrada da molte vnitade. et al meno da do vnitade. come e. 2. el quale e lo primo e menore numero: che se truoua. La vnitade e quella cosa: da la quale ogni cosa si ditta vna. Segondario sapi: che se truoua numeri de tre maniere. El primo se chiama numero simplice. lal tro numero articulo. El terzo se chiama numero

Libro de Marco Beltrami

Fig. 3.1 Reproduction of the first page of the *Arithmetica di Treviso* (1478), published by D.E. Smith (1948), p. 4

Rechenbuch in 1483 and Johannes Widmann's *Rechenbuch für die Kauffmannschaft* in 1489, printed in Eger (Bohemia), both in German.[1]

It is highly significant that these printed books were written in the vernacular; textbooks – at least in that period – made no distinction between specialists and laymen; they had rather the aim, for the first time, to make knowledge how to calculate accessible to the general public.[2]

A considerable production of applied mercantile books can most likely be established for all European countries. Studies for the following countries show the extension of this new international degree of textbook production for this kind of mathematical practice.

[1] These first arithmetic textbooks are accessible in reprints, resp. translations: the Treviso arithmetic is translated in Swetz (1987); the *Bamberger Rechenbuch* as a reprint by Eberhard Schröder in 1988, and Widmann's book in a careful edition by Barbara Gärtner (2000).

[2] The book *Rara Arithmetica* by David E. Smith presents an excellent commented documentation of arithmetic books printed until 1601.

– *Germany*. This country is the one best studied for its mercantile arithmetic textbook production and their authors, the *Rechenmeister* – the literal translation of the Italian *maestro d'abbaco*. The reason is that its main representative, Adam Ries (1492–1559) – also known as Riese – became proverbial; effecting a basic arithmetic operation orally, Germans of all social layers used to comment its correctness: "das macht nach Adam Riese" – *this makes according to Adam Riese*. His textbook *Rechnung auff der Linihen und Federn* (1522) was reprinted at least 120 times until the seventeenth century (Fig. 3.2). Originally from Staffelstein, near Bamberg in Southern Germany, he established first a *Rechenschule* (*scuola d'abbaco*) in Erfurt but moved in 1522 to Annaberg, a strongly expending town in Saxony due to silver mining, established there since the end of the fifteenth century. First acting there again as *Rechenmeister* and leading a *Rechenschule*, he became the official and *Rechenmeister* of the mining administration of Annaberg. It is highly revealing that Ries had mercantile arithmetic not as his only focus; he also worked with algebra. In 1524, he finished a voluminous algebra manuscript, entitled *Coß* – the German term for operating with unknowns. Yet, he did not publish it; maybe due to the *Coß* published in 1525 by Christoff Rudolff (1499–1545) – only in 1992 was it published, in a well commented facsimile edition by Hans Wußing and Wolfgang Kaunzner.

Fig. 3.2 Cover of one of the numerous reeditions of the classic textbook by Adam Ries, of 1574

Given his important impact upon valorising arithmetical knowledge, Adam Ries has been traditionally well researched. More recent research has not only focused on his sons, most of them became *Rechenmeister*, too; particularly important was Abraham Ries (1533–1604). He also wrote a *Coss* textbook (Abraham Ries 1999). His extended activities as *Rechenmeister* and algebraist are well documented in the volume (Folkerts and Rüdiger 2020). But, in particular due to the *Adam-Ries-Bund* (Annaberg-Bucholz/Germany), dedicated to commemorate the professional group of *Rechenmeister* in Germany, since the sixteenth century, there has been intense on-going research upon these practitioners. This group stimulates research upon life and work of the numerous *Rechenmeister* who acted in Germany and regularly organises workshops. Hence, its Proceedings provide important sources for this research. To name one such revealing volume: *Rechenmeister und Cossisten der frühen Neuzeit* (1996). Particularly important is the volume (Folkerts and Gebhardt 2009), which documents the works achieved by the Adam-Ries-Bund between 1992 and 2008.

Another *Rechenmeister*, as well-known as Adam Ries, but in a later period, was Johannes Faulhaber (1580–1635). He is known in historiography of mathematics since it is considered whether Descartes had visited him in Ulm and discussed issues of cossist algebra. Two rather recent biographies have been published about him, with contradictory positions (Schneider 1993; Hawlitschek 1995).

Faulhaber's case is really revealing: although being author of textbooks, he intended to keep his specialised knowledge hidden in a revealing manner, not keeping it secret but dissimulating it by using an unparalleled idiosyncratic terminology, and an unnecessary separation of types of problems, endowed with terms and methods of calculation invented by himself (his intention seems to have been to increase his income by attracting disciples to his own private school, teaching them his hidden procedures). Such ambiguity is typical of the transition process from previously predominant private to public forms of teaching, the process becoming even more complicated as there were still no conventions generally agreed upon regarding signs and terminology, a situation that made texts quite difficult to read and understand.

– *Italy*: Italy being the country where this kind of textbooks had developed first within Europe, during the Middle Ages (see Chap. 2), the production of mercantile textbooks continued strongly also in print. Two specialised researchers are Raffaella Franci and Laura Toti Rigatelli; they have published a very instructive synthesis of their research (Franci and Toti Rigatelli 1982). To analyse the mercantile practices in use there, they chose the production of these textbooks in the Tuscan city of Siena, where the textbooks by maestro d'abbaco Dionigi Gori (1510–ca. 1586) were paid for by the city government. Gori was also in charge of the superintendence of the water system and the roads of the Siena state. His manners for the basic operations are shown; then, the metrological systems for measures and currencies are explained. The main issue is operating within these systems for the various mercantile applications, based on the rule of three. There were also sections on entertainment mathematics.

– *France*: Jochen Hoock, a historian, has studied the production of merchant textbooks since the fifteenth century, creating a database for them (see Hoock 1986). This database developed and assumed quite monumental dimensions: on the one hand, the project extended to explore the production of such books throughout Europe and even in the Americas. The first volume indicated as aim of the project to be the directory of books printed throughout the area of European civilisation as textbooks for merchants: textbooks for training, treatises including technical commercial practices (ibid., p. VII).

And on the other hand, the period was later extended to the year 1820. Publication of the identified printed books had reached the year 1700 (see Hoock et al. 1991–2001). As the printed textbooks until about 1600 consisted mostly of commercial arithmetic – in later periods, bookkeeping manuals predominated - this fact, actually universal, allows one to study and compare the applied arithmetic textbooks for at least all the European countries. Already the first volume begins with a careful methodical introduction, explaining the definition and demarcation of the text corpus as well as the intention of the research. This introduction also includes a list of the consulted libraries in Europe and in the USA. The main body of the first volume is a documentation of the textbooks published from 1470 to 1600, alphabetically according to their authors.

Volume 2, for the period from 1600 to 1700, continues the search and documentation according to the principles established in the first volume, with adaptations due to specific technical problems arisen in this period, and due to rapid changes of the content structure of commercial literature. Volume 3, published in 2001 titled *Analyses (1470–1700)*, contains several pertinent chapters studying aspects of these textbook analyses, like publishers' market strategies, variations in the contents of bookkeeping texts, the passage from textbooks to compendia, study on cover graphics in Dutch and German textbooks (Hoock et al. 2001).[3]

France presents for this period a characteristic example of the degenerate triangle between textbook, teacher and student: the highly popular arithmetic textbook by François Barrême (1638–1703), claiming to be studied without a guide, without a teacher:

L'Arithmétique du S^r^ Barrème – Le livre facil, pour aprendre l'Arithmétique de soy mesme, & sans Maistre, par des Methodes si courtes, si claies et si bien ordonnées, qu'il ne s'en est point veu de pareilles, first published in 1672[4] and reprinted many times, even in the nineteenth century. Barrême is likewise proverbial in France as Adam Ries in Germany.

– *The Netherlands*: This country, well-known for its wide-ranging intense commercial activities, also developed a great number of commercial, practical arithmetic textbooks. A revealing contextualisation is given by Dirk Struik's book (Struik 1981). The excellent PhD thesis by Marjolein Kool has analysed in detail the areas of arithmetic dealt with in these textbooks, the enormous multitude of "rules" presented for operating in these areas, the terminologies and notations used, and the didactical procedures for presenting the material. She succeeded in identifying 23 textbooks for the century, from 1500 to 1600, some by anonymous authors, and various with a few reprints. The first textbook is of 1508, an anonymous one:

– *Die maniere om te leeren cyffren na die gerechte consten Algorismi. Int gheheele ende int gebroken*. [The manner of learning numbering, after that righteous art Algorismi. For entire and for broken]

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3 The project, has published so far three of the six planned volumes. Volume 4, covering the period 1700–1760, was announced for 2020.

4 The year of this first edition is almost never indicated. Hoock's excellent bibliography gives the data (Hoock 1991, p. 21, 23).

And the last one, of 1600, by Jacques van der Schuere:

– *Arithmetica, Oft Reken-const, Verchiert met veel schoone Exempelen, seer nut voor alle Cooplieden, Facteurs, Cassiers, Ontfanghers, etc*. [Arithmetic, or Art of Reckoning, enriched with many beautiful examples, very useful for all merchants, facteurs, cashiers, beginners, etc.]

Several of these textbooks show very nice diagrams, indicating the steps of the operational procedures. Here is such an example for the division procedure, in the form of a sailing ship (Fig. 3.3). An analysis for the following seventeenth century is given in Chap. 1 of van Maanen (1987).

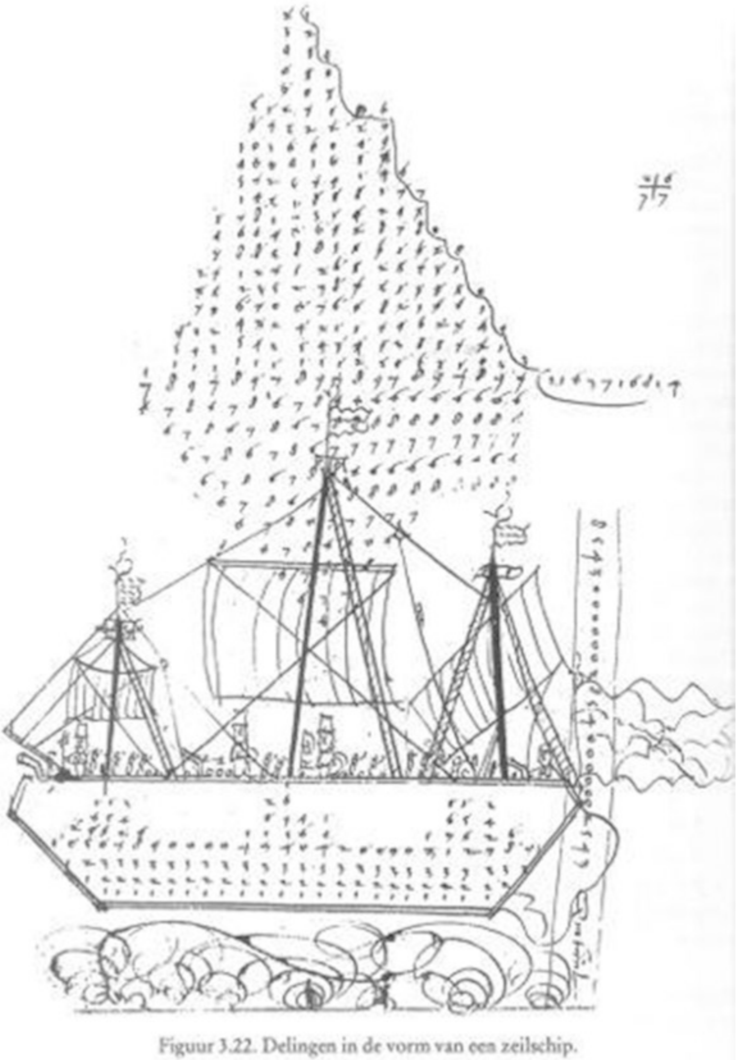


Fig. 3.3 Division procedure, from a 1584 manuscript (Kool 1999, p. 88)

– *England*: Regarding England, there is an excellent documentation of this group, aptly coined mathematical practitioners, for the sixteenth and seventeenth centuries (Taylor 1954) and a continuation for the eighteenth century (Taylor 1966). The textbook and manual production by a wide range of professionals – instrument makers, surveyors, artisans, accountants, arithmetic teachers – is documented in biographic and bibliographic information.
– *Spain*: During the last years, numerous investigations have focused on early Spanish printed mercantile books. Important results were obtained first by Vicent Salavert in 1990: his assessment of printed books in Spain from 1482 to 1600 yielded a total of 43 commercial arithmetic books, published by 35 authors and some published various times so that there were 77 editions (Salavert 1990). Some characteristic examples for the sixteenth century are the following:

- "Tractado sutilissimo de Arismetica y de Geometria", by Juan de Ortega, first edition of 1512;
- "Sumario breve de la practica de arithmetica de todo el curso del arte mercantivol", by Juan Andrés, 1515;
- "Practica mercantivol", by Joan Ventallol, 1521;
- "Tratado de cuentas", by Diego del Castillo, 1522;
- "Compendio de los números y proporciones", by Pedro Melero, 1535;
- "Arte breve y muy provechoso de cuenta castellana y Arismetica", by Juan Gutiérrez de Gualda, 1539;
- "Arithmetica practica y especulativa", by Juan Pérez de Moya, 1562 (Madrid et al. 2019, S. 4).

The structure of these books is quite similar, in general: introducing the number system and operating with numbers and with fractions. The main issue is to deal with the rule of three or with proportions. A particularly close analysis has been elaborated by Maria José Madrid, Alexander Maz-Machado, Carmen López, and Carmen León-Mantero (2019). I am quoting here their main results, which also apply well for textbooks of other European countries:

> None of the books include a definition of number or quantity. All the authors begin their books with a (more or less) detailed explanation about the positional numeral system.
>
> After explaining the numeral positional system, these books present several chapters about the four operations: addition, subtraction, multiplication and division. The difficulty of the multiplication algorithm, maybe related to the difficulty of memorizing the times tables, is reflected in the inclusion of numerous examples of multiplication, rules to remember the tables and even different multiplication algorithms, which are not proved.
>
> With the exception of Gutierrez's, all the other books included fractions or operations with them (change fractions into fractions with common denominators; addition, subtraction, multiplication and division of fractions). Among the authors who include a definition of this concept, it is mostly widespread to consider that a fraction, frequently referred to as a broken number, is a part of the whole number or a thing that is not whole.

> None of the books includes any definition of proportion. However, all of them include, to a greater or lesser extent, exercises about the rule of three or rule of company. Despite the fact that these rules carry implicit the notion of proportion, it is not explained by the authors.
>
> The problems included are exercises on the rule of three, both directly and inversely proportional, simple and compound; on proportional distributions; on percentages, and on false position methods (Madrid et al. 2019, p. 9 f.)

Besides the intense production of textbooks for commercial arithmetic, in Spain they also started publishing textbooks developing algebra, in the mid-sixteenth century. Using Luca Pacioli's term *Arte Magiore*, algebra was called there *Arte Mayor*. Quite recent research has shown that this development was initiated by Marco Aurel's textbook. Born around 1520, he was not only a German immigrant in Valencia but also published a markedly clear direct transmission of German cossist algebra, using Rudolff's textbook through often literal translations (Romero-Vallhonesta and Massa-Esteve 2018, p. 79 ff.). His textbook *Libro primero de Arithmetica Algebratica …* (1552) was soon followed by *Compendio de la Cosa o Arte Mayor* (1558), by Juan Pérez de Moya (1513–1596), and *Arithmética* (1564), by Antic Roca (about 1500–1580). Both were based largely on Aurel's conceptions, but Pérez de Moya transferred the Rudolff-Aurel gothic signs into Latin while Roca used more those Pacioli had used (Massa-Esteve 2012, p. 111 f.).

– *Portugal*. The production of arithmetic textbooks in Portugal has also been intensely researched. The reference for these studies is the two-volume work by A. A. Marques de Almeida (1986). It identifies arithmetic as a social and cultural phenomenon, revealing a transformation of the society, using this kind of knowledge to meet social needs and develop economic practices. The broad dissemination of the commercial arithmetic textbooks is understood as revealing a modern mentality, emphasising innovation, practicing measuring, and accepting exactness. Five of such textbooks were published between 1519 and 1624.

The first one, by Gaspar Nicolás (1519), is also the one most often reprinted – ten times, until 1716: *Tratado de pratica Darismetyca*. In addition to the basic four operations using Arabic number signs, there are extended expositions of proportion rules, a few problems of geometry – determining length and area - and an appendix about alloying silver. Despite intense research by many historians, almost nothing is known about Nicolás's biography and this also applies to the other four authors (Marques de Almeida 1994, pp. 54 ff.).

The other authors and their books were:

– Ruy Mendes: *Pratica darismética* (1540);
– Bento Fernandes: *Tratado da arte de arismetica novamente composto e ordenado* (1555), the second edition of a 1541 version entitled *Arte de aritmetica dedicada ao Infante D. Luis*, of which, however, no copy has been identified so far;

– Gaspar Cardoso de Sequeria: Da Arismetica, com várias curiosidades a ela pertencentes (1612); and
– Afonso de Villafanhe Guiral e Pacheco: *Flor de Arismetica necessaria* (1624). (Marques de Almeida 1986, 1994).

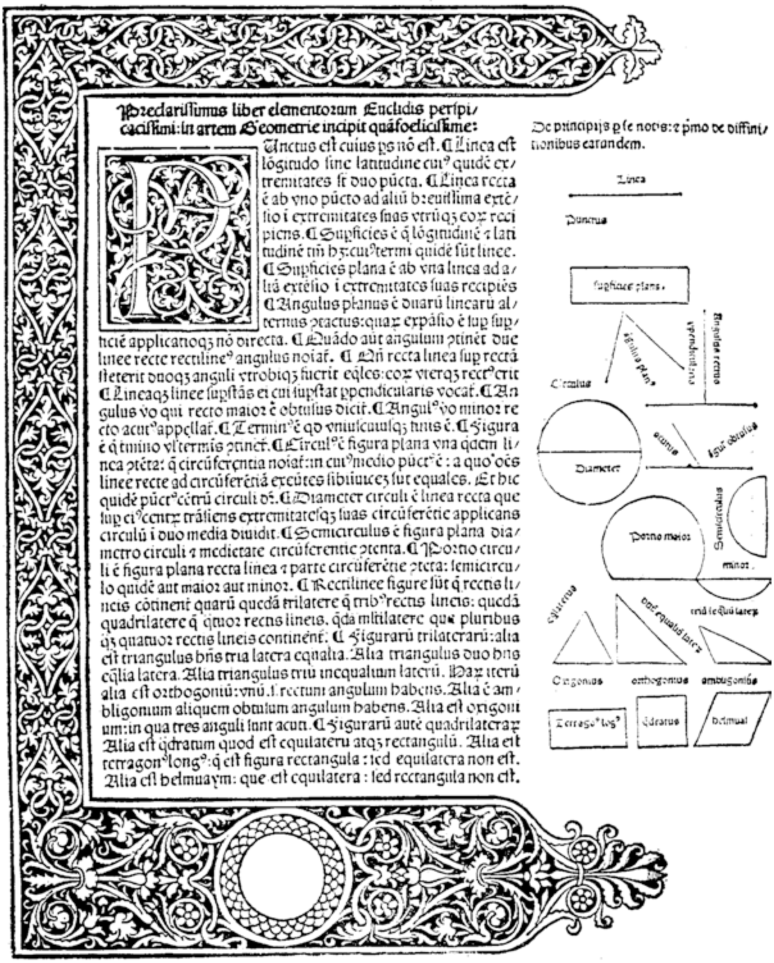
José Manuel Matos's study (2007) focuses in particular on Nicolàs's textbook and on the use of the commercial textbooks in teaching.

One can rightly assume that an analogous movement for the publication of practical arithmetic textbooks emerged in the Scandinavian countries, too. Kristín Bjarnadottir studied the first such textbook in Iceland, *Arithmetica Islandica* of 1716, and showed that it is a free translation of a Danish textbook, the *Arithmetica Danica* of 1649 (Bjarnadottir 2018, p. 232).

The first textbook of that type printed in Russia was *Arifmetika, sirech' nauka chislitel'naya, s raznykh dialektov na slovenskiĭ yazyk perevedenaya, i vo edino sobrana, i na dve knigi razdelena* [Arithmetic, or learning of numbering, translated into Slavic language from different dialects, assembled and divided into two books], published by Leontiĭ Filippovich Magnitskiĭ (1669–1739) in 1703, a textbook which had a lasting effect. It was an adaptation of one of the most successful Dutch textbooks, *Arithmetica oft reken-kunst* by Jacob van der Schuere (1576–ca. 1643). A recent analysis is by Viktor Freiman and Alexei Volkov (Freiman and Volkov 2015).

3.3 First Printed Versions of Euclid's *Elements*

In addition to this extensive publication of practical arithmetic textbooks, the use of textbooks in universities has undergone several critical changes. As Humanism (from about the middle of the fifteenth century) encouraged interest in the classical age of antiquity the original Greek and Roman texts began to be edited and printed, being used in university teaching. This also resulted in the first printed editions of mathematics: the first edition of Euclid's *Elements* in Latin occurred in 1482 (Fig. 3.4), and the first edition in Greek in 1533. From 1482 onwards, a steadily increasing stream of editions of the *Elements* were printed, and they can be considered a bestseller of Humanism. These editions were followed by vernacular translations between 1543 and 1564 into Italian, English, German, and French. Other languages later followed, but it is unclear how much these books were used in university teaching (see Schreiber 1987).


Preclarissimus liber elementorum Euclidis perspicacissimi: in artem Geometrie incipit quãfoelicissime:

De principijs p se notis: e pmo de diffinitionibus earundem.

Fig. 3.4 Reproduction of the first page of the first printed version of Euclid's *Elements* (1482), published by D. E. Smith (1951, p. 250)

The editors of these Euclid editions proved to be highly dedicated to "improving" the text in *Elements*, quite analogous to the practice of commenting or making addition to a textbook in earlier periods since Antiquity. The focus of these numerous editors, until the eighteenth century, was the axiomatic system. As Vincenzo de Risi has shown in his excellent study (de Risi 2014), either there were alterations in the formulation of the axioms or the editors added new axioms so that even quite extensive axiom systems substituted the original ones.

One event turned out to be decisive for this strong dissemination of Euclid's *Elements*: the Society of Jesus, founded in 1534, adopted Euclid's *Elements* in 1552 for the teaching of mathematics in its Colleges. Perhaps it was this fact that sparked Petrus Ramus's (Pierre de la Ramée) campaign against Euclid, which gradually resulted in the establishment of remarkable new textbook traditions in France.

Ramus (1515–1572) was murdered in the Saint Bartholomew massacre, not as an accidental victim, but because he was actively involved in the fight against Aristotelianism and Scholasticism within the universities. This made him one of the most ardent enemies of the Jesuits, who had renewed Aristotelianism; in their later *Ratio Studiorum* (1599), the uniform curriculum for all their colleges in whatever country, the teaching of mathematics was included as a subject in in the philosophy course (the last years of their college). This meant that the Jesuits understood mathematics as part of their particular brand of philosophy. As Ramus himself had published an edition of Euclid in 1541, it can be rightly assumed that the Jesuits' later choice of Euclid induced him to criticise this classical author.

One of Ramus's main achievements was his criticism of Euclid's *Elements*: not only of particular propositions or the accuracy or rigour of certain demonstrations, but much more fundamentally it was the methodology of the *Elements* which constituted his key target of criticism. Euclid's work, he said, had always been estimated to be above criticism, and hallowed all over the world, for almost 2000 years (Ramus 1569, p. 74). In Ramus's view, the *Elements* were not, as traditionally thought, the universal model for rigorous reasoning and logical deduction, but this book rather revealed a lack of natural, methodical order. Ramus, therefore, on the one hand developed rules for methodical thinking, and on the other hand proposed an entirely different order and architecture for mathematics: it should begin with the general, and the general was, in Ramus's understanding, not geometry but arithmetic. Furthermore, arithmetic and geometry should first be treated separately and then combined. Ramus understood arithmetic to be the more fundamental ("prius"), more general and simpler discipline of mathematics, whereas he saw geometry "by its nature" as a particular discipline based on arithmetic, while Euclid, by contrast, had begun his *Elements* with geometry (ibid., 97). Even in geometry, Euclid had not adhered to the basic rule of developing the general before developing the particular (ibid., 98–99). Therefore, Ramus seems to have been the first humanist to reflect on the methods and structure of textbooks.

What began in France with Ramus was a dialectical development of fundamental importance for the later evolution of mathematics: the reception of Euclid's book in its original language led, in the context of Humanism, to a reflection on the foundations of mathematics, on its methodology and architecture; and that – gradually – led to the conceptual revolution of Mathematics around 1800. The critique of Euclid, initially undertaken in France as a "Sonderweg", a particular trajectory, restricted to the context of this nation, metamorphosed into an effort on mathematics, which in general endeavoured to increase rigour in order to meet the global methodological demand. Here, once again, textbooks provided a productive stimulus to the development of mathematics.

3.4 Printing Diagrams

The printing of mathematical books implied additional challenges for the new profession of printers. While it was necessary, for printing normal text, to provide a stock of movable letters, printing a mathematical text afforded, firstly, to print numbers – and cast numbers like movable letters constituted no specific problem. It might have already been a bit more difficult to have all the signs for operations, for roots, etc. Probably, printers assisted by mathematicians will have succeeded in casting these signs, too. But how did these craftsmen succeed in printing geometrical diagrams? As the Fig. 3.4 above shows, the lines of triangles, rectangles, circles, etc. do not appear to be very clear-cut and exact. And they are printed within the pages with the text – different from the pages with diagrams printed separately, at the end of the book. How were these first printings of geometrical diagrams realised?

The questions of printing early mathematics, and in particular of printing diagrams have only rarely been addressed. One of the few pertinent studies is by Robin Rider. She emphasised the continuity between the traditional mode of writing manuscripts and the new mode of printing: “In the literature on history of the book, it is a commonplace that early printers sought to make mechanical imitations of manuscripts” (Rider 1993, p. 93). Actually, regarding diagrams, the continuity is with the former primitive manners of printing: they consisted of reproducing a sheet by having text and figures engraved in a woodcut. For geometric figures, printers would now reserve some space on the plate for inserting one or more woodcuts for the diagrams. Rider comments the geometric diagrams in the first Euclid printing, in 1482, achieved by the experienced printer Ethard Ratdolt in Venice:

> Reproduction of geometrical diagrams proved a particular challenge to Ratdolt. Without them, he knew (and said), the text was unintelligible. He met the challenge by placing his woodcut diagrams in the margins outside the text, and boasted about his innovation. […] Ratdolt's Euclid, like manuscript versions that preceded it, labeled features of diagrams with letters and words scattered under/over/inside the figures, sometimes at a considerable angle to the type page itself. To sustain this manuscript practice in print, Ratdolt mixed metal type and woodcuts on a page, and labored to produce, with one pull of the press, equally dark and distinct images from cast metal and carved wood (Rider 1993, p. 96).

The form of diagrams printed on separate sheets began, according to Rider, by the end of the fifteenth century, by copperplate engraving.

> Though engravings were in general more expensive than woodcuts and wore out more quickly, it did not necessarily require a master of the engraving art to produce a creditable, meaningful, and even complicated geometrical diagram on a copper plate. Similar effects were not so easy to achieve (and sustain) when woodcuts were used. Use of less than expert woodcuts from the outset or the practice of using old or worn blocks yielded diagrams of distinctly uneven quality (ibid., p. 98).

These copperplate engravings are also known as intaglio printing – a technique in which the image is incised into a surface and the incised line or sunken area holds the ink. I am extensively quoting this technology to show the complexity of this new printing procedure:

> Because production of intaglio prints required a separate, different press from that used to print from both metal type and woodcuts, it was generally easier to segregate engravings from type, often by filling a single plate with several engraved diagrams. This is not to say it was impossible to print the same sheet on two presses and thus to mix metal type and engravings; it was, however, technically difficult to align print and plate properly, and the expense involved in such careful presswork often proved prohibitive. The wrapping of text around relevant diagrams typical of technical books with woodcuts […] was thus relatively uncommon in works illustrated with engravings. Indeed, the publishing practice of printing all engraved figures on separate (often foldout) pages, often far removed from the relevant text, reinforced a trend in mathematics itself: over the course of the 17th and 18th centuries, analysis of mathematical relationships as expressed in the formalism of algebraic language grew ever more distant from geometric context as depicted in diagrams, just as all diagrams were relegated to the back of the book (ibid., p. 99).

Albrecht Dürer used to be named as an outstanding artist introducing this intaglio technique, by 1500. A Wikipedia site refers to a decline of the woodcut techniques around 1550 and to intaglio techniques dominating thereafter. For Rider, too, geometric diagrams in mathematics books were printed on separate sheets, at the end of the book, through the seventeenth and eighteenth century. It seemed, therefore, that it was only during the nineteenth century that diagrams returned to the interior of the text, accompanying their related text passages – and this by some technological change in printing. Actually, so far, I could not find references to these changes. Moreover, I became aware that a complete change to tables with copperplate engravings did not occur during the sixteenth century. There are numerous textbooks, printed in France, England, and Germany, that still contain diagrams within their respective text passages. Figure 3.5 gives an example from Antoine Arnauld's 1667 textbook.

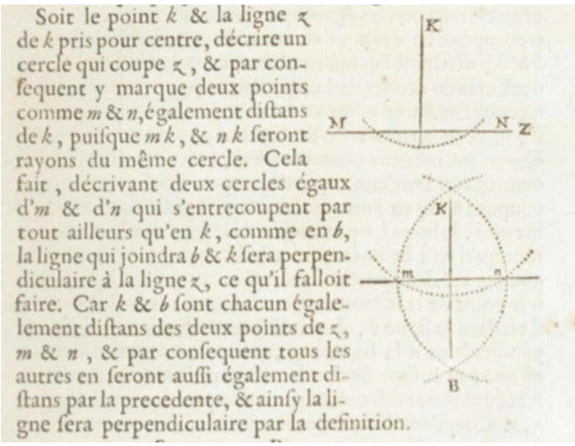

Soit le point *k* & la ligne *z*
de *k* pris pour centre, décrire un
cercle qui coupe *z*, & par con-
ſequent y marque deux points
comme *m* & *n*, également diſtans
de *k*, puiſque *mk*, & *nk* ſeront
rayons du même cercle. Cela
fait, décrivant deux cercles égaux
d'*m* & d'*n* qui s'entrecoupent par
tout ailleurs qu'en *k*, comme en *b*,
la ligne qui joindra *b* & *k* ſera perpen-
diculaire à la ligne *z*, ce qu'il falloit
faire. Car *k* & *b* ſont chacun égale-
lement diſtans des deux points de *z*,
m & *n*, & par conſequent tous les
autres en ſeront auſſi également di-
ſtans par la precedente, & ainſy la li-
gne ſera perpendiculaire par la definition.

Fig. 3.5 In-text diagram, example from Arnauld (1667, p. 89)

More examples will be mentioned in the following chapter. The issue of diagram printing is not well studied pertaining to the seventeenth and eighteenth centuries.[5]

There is another open question regarding how teachers used textbooks in their classroom, in Pre-Modern Times: How did they visualise calculations or draw diagrams? Today, blackboards are commonly thought to have always existed as teaching devices for textbook use. Actually, the history of this innovation is not well studied. It seems that its introduction occurred in the late-eighteenth century and that it was predated by the small plates of slate for school beginners. There is evidence that professors used blackboards in 1795 during their lectures at the famous *École Normale* of the year III, the first higher learning institution for teacher education (Julia 2016a, p. 364). It was distinguished for the enormous number of students being taught simultaneously in the big amphitheatre: about 1.400 (see Chap. 4). The last volume of the series of studies about this *École Normale* revealed, thanks to studying all preserved files on any detail that the caretaker of the lecture hall was in charge of buying regularly, for the mathematics lessons, pieces of chalk, in packets of dozen, and sponges "necessary for the demonstrations on the blackboard" (Julia 2016a, p. 143).

However, there is evidence of a very peculiar and apparently unique teaching device for using diagrams in geometry lessons. Since 1930, some *azulejos* with geometric diagrams were sold by antique dealers; nowadays, 26 of such tiles are preserved – the majority in the Museu Nacional de Machado de Castro, in Lisbon. *Azulejos* are typical Portuguese blue-coloured tiles. In the late 1990s, their origin began to be researched. It was then proved that there had been a "aula da esfera", a mathematics classroom – literally, however, "astronomy classroom" – in *Colégio de Santo Antão,* in Lisbon, the main Jesuit college in Portugal. And its walls had served for exposing *azulejos* with geometrical and astronomical diagrams. They are dated to shortly after 1692. It had even been possible to determine for which textbook this device had been constructed: it had been the Euclid edition by the Belgian Jesuit André Tacquet (1612–1660). His edition, first published in 1654, but reedited numerous times and translated into various languages until the eighteenth century, had the special feature to include material from

[5] The book edited by Cynthia Hay (1988) does not discuss printing techniques, despite its promising title – *Mathematics from Manuscript to Print, 1300 to 1600.*

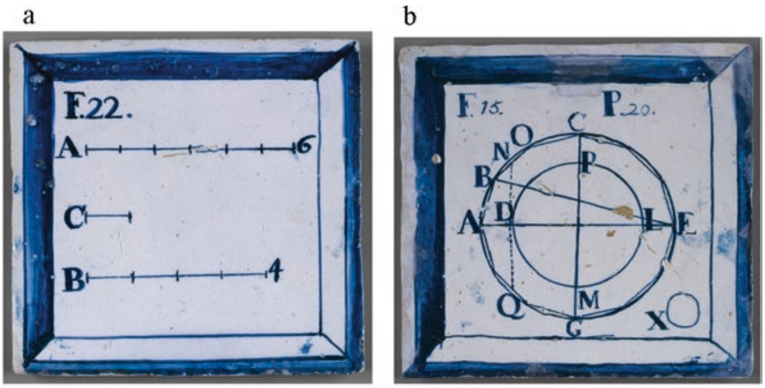


Fig. 3.6 *Azulejos* from the *Aula da Esfera* – (**a**) Lemma 2, a comment to Euclid's Book V by Tacquet. (**b**) Proposition 20 in Tacquet's part on Archimedes (Azulejos que ensinam)

Archimedes, as already indicated in the title: *accedunt selecta ex Archimede theoremata* (Tacquet 1654). And various tiles referred to theorems from Tacquet's edition (Fig. 3.6a, b; see Leitão 2007).

Chapter 4
Differentiation of Textbook Development During Pre-modern Times

4.1 Textbooks for New Publics and for New Areas

The printing technique had enabled mathematical knowledge to be available to the social strata in European states which before had not had access even to literacy. The emerging capitalist societies were depending on such dissemination of practical competencies. But also the traditional very small groups who had access to functions in administration, health, and clerical services extended, due to the first establishments of educational structures – in particular by differentiating the former general institutions (*Hohe Schule, Archi-ginnasio, Gymnasium Omnium Disciplinarum*) into secondary schools (college, *Gymnasium*, etc.) and in higher education universities. Due to this differentiation, also textbooks were became differentiated during this period: from addressing at first a general public to becoming textbooks for secondary schools, resp. for higher learning, and also for technical formation. Textbooks for primary schools essentially date from the French Revolution on. Moreover, the ensuing enormous development of the mathematical areas led to the establishment of ever new disciplines, so that the originally more embracing and general textbooks became differentiated into textbooks for specific branches and disciplines.

4.2 The Growth of Algebra Textbooks

It is not well known that the first specialisation of mathematics textbooks in Pre-Modern Times, after the general wave of mercantile textbooks, were textbooks on algebra. In fact, a considerable number of medieval European arithmetic textbooks had exposed some algebra issues – not astonishingly, given that they were written based on transmissions of Islamic textbooks (see Sect. 2.7). For instance, the

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_4

recreational problems proposed in various of these medieval textbooks afforded to solve systems of (linear) equations.

One might say the manuscript *La Triparty en la science des nombres*, written in Provençal, in 1484, by Nicolas Chuquet (ca. 1445–1488), was the first genuine algebra textbook. In its first part, Chuquet dealt with numbers (including negative numbers), in the second with roots, and in the third he was devoted to solving first-degree and second-degree equations. Chuquet did not use a sign for the unknowns (he did not even have a specific term for them, using "numbers") but used signs for roots, and he used exponents for higher roots.

While this work remained rather unknown, a strong development of algebra was achieved by the German-based cossist algebra – 'coss' from the Italian word 'cosa' is the translation of the Arabic expression 'shay' for the unknown – which was based on the transmission of the Maghrebian symbolism and had developed it further. The powers of the unknowns were formed multiplicatively and the signs for higher powers were for some of them multiples of the first three powers, while new signs were introduced for others of the higher powers (see Fig. 4.1).

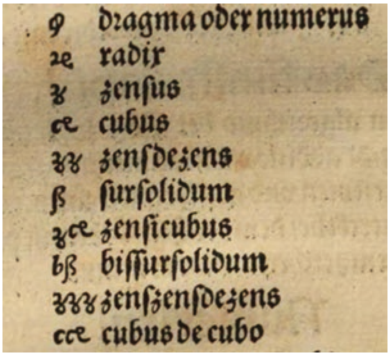


Fig. 4.1 Rudolff (1525), p. 24v

The first of such books, *Die Coß* (1525), was written in German, by Christoff Rudolff (ca. 1499–1543), printed with Gothic letters no longer familiar today and likewise unusual in other European countries. The lengthy title emphasised even that the book could be studied without a teacher, constituting thus probably the first case of the "degenerate" textbook triangle: The first lines of the title claim: *An elegant and beautiful calculation of the artistic rules of algebra – as commonly known as the Coß. All so faithfully presented that even alone, by diligent reading without any oral teaching, it can be understood* (Rudolff 1525, cover). With the exception of the sign for the numbers, symbolising a zero, the first four powers of the unknown are symbolised by the first letter of the term (in Latin, or in a German adaptation of the Italian at the time), while the signs for the higher powers are either new signs or combinations between these new signs and those of the first four. Rudolff worked with up to the ninth power of an unknown. His book focused completely on developing the various parts of algebra, presented as a series of "rules" accompanied by examples. There are no recreational problems.

The second important work was the *Arithmetica Integra* (1544), in Latin, by Michael Stifel (ca. 1487–1567). Originally an Augustinian monk, Stifel became a pastor in various parishes and studied mathematics in Wittenberg. Later he was a mathematics lecturer at the universities of Königsberg and Jena. The book, which came about when he was a pastor in the small town of Holzdorf, shows that this algebra was not developed in isolation. It referred to several works by Cardano. And it made two important innovations; it has defined the number of powers of the unknown beyond 9 in an arbitrarily general way and thus freed the concept of the unknown from geometric connotations and algebraised it. Stifel specified the sequence of symbols up to the 16th power and declared it to be unlimited with "deinceps in infinitum".

His book reveals an enormous conceptual richness. In his first part, Stifel introduced "abstract numbers" and rational numbers as fractions, as well as the basic operations with them. He showed negative numbers ("numeri absurdi") in a graph as a continuation of the series of positive numbers beyond zero (Stifel 1544, p. 249v). Arithmetic and geometric progressions, proportions, figured numbers, square, and cube roots were further sections. He also showed that extracting roots could in principle be continued at will; he gave examples on the sixth, seventh, eighth, ninth, and tenth root. The second part contains a detailed presentation of operating with irrational numbers.

The other important innovation was that Stifel introduced to operate with more than one unknown. In the Maghrebian practice of symbolism, algebra had remained restricted to operate with just one unknown. Due to missing signs, working with more than one unknown afforded some rhetorical expedient. Rudolff had operated in his *Coß* with a second unknown; he rhetorically referred to it as "quantitet" and operated jointly with the sign x for the first unknown., and "quantitet" for a second unknown (Rudolff 1525, S. 118r). Stifel, however, was the first who not only introduced symbols for unknowns other than the first but also did not limit their number. He mentioned that Rudolff and Cardano had introduced a second unknown with the term *quantitas* (Stifel 1544, p. 252r). Stifel described the first unknown as "numerus absconditus" and the other unknowns as "secundi radices", second solutions (Stifel 1544, p. 251v). Stifel explained: if another unknown occurs in a problem after an unknown named $1x$, then it is named $1A$, or more precisely as $1Ax$. One must distinguish the second solutions from the first by other signs. He also understood third, fourth, fifth, etc. solutions as "secundi radices". For didactic reasons, all these other unknowns are called second solutions in relation to the first solution (unknowns). All the usual operations can be carried out with them (ibid.). For these other unknowns, Stifel no longer used the cossist Gothic letters as symbols but the Latin (upper) letters of the alphabet: again, without a limit in their number (Fig. 4.2.):

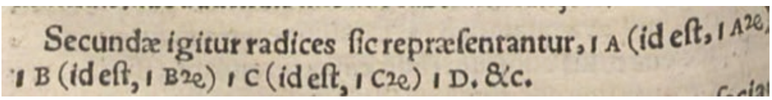

Secundæ igitur radices sic repræsentantur, 1 A (id est, 1 A𝔵,
1 B (id est, 1 B𝔵) 1 C (id est, 1 C𝔵) 1 D. &c.

Fig. 4.2 Stifel (1544), S. 251v

Ninety out of the more than 600 pages discuss examples, yet not vested as recreational problems; they address issues of the respective chapter in mathematical terms. In 1553, Stifel published a revised and expanded version of Rudolff's *Coß* from 1525. The considerable extension – from 416 pages to 986 – consisted of explanations and details added to each of the chapters.

Another textbook of cossist algebra is generally better known than the books by Rudolff and Stifel: that by Christopher Clavius (1538–1612), first published in 1608 in Rome. This is due, on the one hand, to Clavius's international role as the leading mathematician of the Jesuits, and as the reformer of the Julian calendar (Clavius 1608). On the other hand, it became well known ever since there was a general conviction in historiography that Descartes had learned algebra from Clavius's textbook, at the Jesuit college La Flèche. Closer analysis reveals this as one more of the many myths regarding mathematics: algebra was not part of the very restricted mathematics curriculum of the Jesuit *Ratio Studiorum*. And according to the catalogue of the college library in 1777, there were few volumes by Clavius in the library, in particular, of his Euclid edition and of the calendar reform (Clavius 1603), but none were his algebra book (Baudry 2014). Since there is no clear evidence of the years when Descartes was a student at that college, one cannot know for sure who was the Jesuit who taught the few months of mathematics within the philosophy grades at La Flèche; but the most probable is a young Jesuit who had just passed some formation with his superior (Romano 1999, p. 383). Descartes could well have learnt cossist algebra from other textbooks, even from French ones (see below).

Moreover, Clavius's book has not contributed to advance algebra Clavius (1608). Peter Treutlein, one of the few who have studied the German cossist algebra more closely, has characterised this book as a weak "imitation" of Stifel's work, with many direct adaptations without any reference to Stifel (Treutlein 1879, p. 21). There was only a brief three-page section in his book on multiple unknowns, in which he adopted Stifel's innovation as *radices secundae* (without citing the source); however, no application was made of this in the rest of the book. Over half of the 380 pages were problem tasks, in the sense of recreational mathematics; various unknowns sought were expressed there in rhetorical terms.

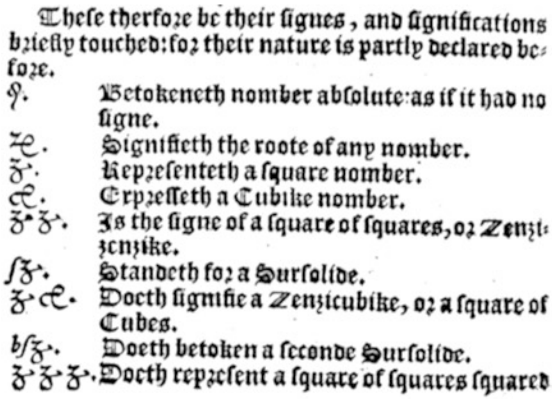
In Italy, where research papers on algebraic problems had already been published by Lodovico Ferrari (1522–1565) and Niccolò Tartaglla (ca. 1499–1567), two textbooks on algebra were published in that period: the *Ars Magna* (1545) by Girolamo Cardano (1501–1576) with solutions of third- and fourth-degree equations, and the *L'Algebra* (1572) by Raffaele Bombelli (1526–1572), where Diophantus's manuscripts had been used for the first time. Based on the proper Italian algebraic tradition, one does not find there a reception of the cossist algebra. In other European countries, in particular, in France, there was an intense impact.

The most direct and immediate reception occurred in France, by Jacques Le Peletier du Mans (1517–1582), in his book *L'Algèbre* (1554). Already in his preface, he emphasised the contributions by Rudolff, Riese, Stifel and Johann Scheubel, and declared to have taken Stifel's "cossist numbers" as the basis for his algebra (Peletier 1554, p. 4). He used in particular the cossist signs for exponents of numbers and quantities. He devoted several chapters to the new conception of *secondes racines*; as he underlined, he did not use their rhetorical terms but the signs as introduced by Stifel due to their facility and better applicability (ibid., p. 96). The following quote shows how he introduced the signs for the second, third and fourth unknown:

> Nous mettrons donq auec lui, pour 1 seconde Racine, 1A; pour 1 tierce Racine 1B; pour 1 quarte Racine, 1C: c'ét a dire, 1A ℞, ou 1 deusieme ℞ : 1B ℞, ou 1 tierce ℞ , etc. (ibid., p. 97).

Another strong impact of the German cossist algebra is shown by Ramus's *Algebra*. In its first edition, published in 1560, and in its extended posthumous edition, in 1586, Ramus also made the cossist sign conception to constitute his basic conception. Yet, he had transformed the Gothic letters into Latin ones, facilitating the use of signs for the unknowns. His Latin letters indicated geometrical meanings of the respective powers of the unknown; for instance: l = latus; q = quadratum; c = cubus; bq = biquadratum – for the first, second, third and fourth power, respectively (Ramus 1586, p. 328). Strangely, however, Ramus did not introduce and use multiple unknowns – neither in the first nor in the second edition.[1]

The reception of Rudolff's cossist algebra in Spain, by Marco Aurel (1552) has already been mentioned (Sect. 3.2). A likewise immediate reception is evidenced in England by Robert Recorde (ca. 1510–1558). After his practical arithmetic textbook, *The Ground of Artes*, first printed in 1543 and reprinted many times, until 1699, he published as a second volume *The Whetstone of Witte* (1557), a textbook on algebra. It was entirely based on cossist algebra (see Fig. 4.3).



These therfoꝛe be their ſignes, and ſignifications bꝛiefly touched: foꝛ their nature is partly declared befoꝛe.

ʒ. Betokeneth nomber abſolute: as if it had no ſigne.
℞. Signifieth the roote of any nomber.
ʒ. Repꝛeſenteth a ſquare nomber.
ꝭ. Erpꝛeſſeth a Cubike nomber.
ʒʒ. Is the ſigne of a ſquare of ſquares, oꝛ Zenzizenzike.
ſʒ. Standeth foꝛ a Surſolide.
ʒꝭ. Doeth ſignifie a Zenzicubike, oꝛ a ſquare of Cubes.
bſʒ. Doeth betoken a ſeconde Surſolide.
ʒʒʒ. Doeth repꝛeſent a ſquare of ſquares ſquared

Fig. 4.3 The cossist signs in Recorde's Whetstone (1557), p. 148

[1] Loget has analysed the different printing techniques in France for printing algebraic symbols, in particular comparing Le Peletier's and Ramus's printed algebraic symbols (Loget 2012).

Analogous reception is evidenced in The Netherlands, in particular in the same period by the textbook, *Arithmétique seconde*, by Valentin Mennher, in 1556 (see Meskens 2010, p. 129); the title means "algebra".

Yet, an independent strand of algebra was developed by François Viète (1540–1603), a self-taught mathematician in France. His research publications on algebra were received, in particular, in England by Thomas Harriot (1560–1621), who developed abstract algebra. Friends of him arranged his manuscript sheets for publishing an algebra textbook in 1630 which, although somewhat reducing Harriot's generalising approaches, instigated a great number of algebra textbooks in Britain, from Willlam Oughtred (1574–1660) to John Pell (1611–1685), and John Wallis (1616–1703).

Rider's 171-page bibliography of algebra books published between 1500 and 1800 is an impressive record of the enormous extension, which the development of algebra had effected within mathematics, since 1500 and, in particular, by textbook productions in many European countries, especially in Britain (Rider 1982).

4.3 General Textbooks for Mathematics

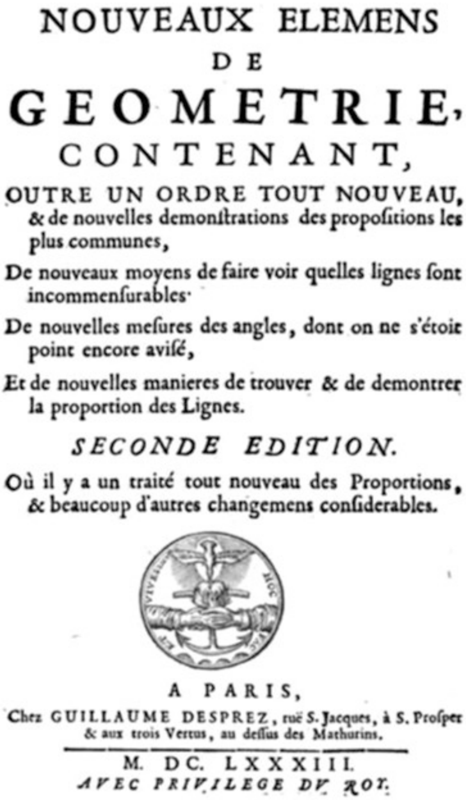
Contrary to what one might have expected, the first textbooks for a general public were not those covering the basic known areas of mathematics, but textbooks exposing this special and new area of mathematics: algebra – which was an offspring of the rise of commercial arithmetic textbooks.

According to its title, Luca Pacioli's *Summa de Arithmetica, geometria, proportioni: et proportionalita* (1494) would have been the first such general mathematical treatise. Although exposing more areas of mathematics, as the books of the mercantile tradition did, the table of contents already showed that the focus was still merchants' necessities.

It is due to the seventeenth century that textbooks for a general public emerged, covering the basic areas of mathematics. It seems that the first of such books was the publication in six volumes of Pierre Hérigone's (1580? – ca. 1643) *Cursus Mathematici – Cours Mathématique*. Actually, it was more of a compendium, containing a very broad range of mathematical works. The first volume, with more than 1.300 pages, contained Euclid's *Elements* (here still in XV books), Euclid's *Data*, five books of Apollonius *Conics*, and a text on the theory of angular sections. The compendium was directed even to an international public: it was published in Latin and in French; pages with pure text were printed in two columns: one in Latin and the other with the French translation. Pages with figures – and there were a great number of in-text figures – were printed sequentially; one paragraph in Latin, then its French translation, and so forth. The second volume contained arithmetic (*pratique, et eccleslastique*) and algebra (*vulgaire et spécieuse*); the third volume contained tables of sines and logarithms, and practical geometry, with fortification, and mechanics. The fourth volume gave the doctrine of the *sphère du monde* (a popular astronomy), mathematical geography and the art of navigation. The fifth volume (1637) contained many applied disciplines: optics, catoptrics, dioptric, perspective, spherical trigonometry, theory of the planets (*tant selon l'hypothese de la terre immobile que mobile*), gnomonic and music. Thus, these volumes constituted the first modern encyclopedia of mathematics – or, maybe better characterised as an anthology. The sixth volume, published posthumously in 1644, dealt again with astronomy and perspective.

The apparently first genuine general textbooks for mathematics were *Nouveaux Élémens de Géométrie*, by Antoine Arnauld (1667) and *Élémens des Mathématiques*, by Jean Prestet (1675). Both textbooks, published by French authors, show pertinent structural elements. At the same time, they caused the first public debate about an issue of foundations of mathematics. And they belong to the specific French way ("Sonderweg") of understanding and teaching mathematics.

Arnauld's textbook can be understood as influenced both by Ramus and Descartes. Descartes's work reveals a direct impact of Ramus's methodological reflections; his well-known eight rules of scientific method breathe the spirit of Ramus's critique voiced against Euclid. And Arnauld's textbook applies the algebraic notations proposed by Descartes for dealing with equations – for instance, naming the unknowns x, y, etc. and the coefficients a, b, etc. Even more profoundly, Arnauld transposed Ramus's methodological critique of Euclid's *Elements* into a new structure of exposing mathematics. Arnauld indicated this critique from the subtitle: "contenant […] un ordre tout nouveau" - containing an entirely new order (Fig. 4.4).


NOUVEAUX ELEMENS
DE
GEOMETRIE,
CONTENANT,
OUTRE UN ORDRE TOUT NOUVEAU,
& de nouvelles demonſtrations des propoſitions les plus communes,
De nouveaux moyens de faire voir quelles lignes ſont incommenſurables·
De nouvelles meſures des angles, dont on ne s'étoit point encore aviſé,
Et de nouvelles manieres de trouver & de demontrer la proportion des Lignes.
SECONDE EDITION.
Où il y a un traité tout nouveau des Proportions, & beaucoup d'autres changemens conſiderables.

A PARIS,
Chez GUILLAUME DESPREZ, ruë S. Jacques, à S. Proſper & aux trois Vertus, au deſſus des Mathurins.
M. DC. LXXXIII.
AVEC PRIVILEGE DU ROY.

Fig. 4.4 Cover of the second edition of Arnauld's book

Antoine Arnauld (1612–1694) was a leading representative of the Jansenist movement, then centred then at the abbey of Port Royal, and as such was expelled from his teaching position at the Sorbonne and persecuted by the Jesuits, just as Ramus had been for being a Protestant. As theologian and philosopher, Arnauld is the author, together with Pierre Nicole, of the other two key textbooks, constituting a new canon for the education of the youth: the famous "Logique" and "Grammaire". Therefore, grammar, logic, and mathematics were the curricular pillars for the *pétites écoles* at Port-Royal. Beyond being an alternative to the *Ratio Studiorum*, the three textbooks were important documents of the French intellectual life, which prepared Enlightenment. The Jesuits succeeded in instigating the government to destroy the abbey. Arnauld lived thereafter in The Netherlands, like Descartes had resolved, too.

While Ramus had written in Latin, Arnauld used French, like Descartes, thus addressing an audience much beyond learned scholars; besides, Arnauld realised a new structure, a new architecture for mathematics. He criticised Euclid's "geometry", saying that this book was more of a mixture of pure geometry and arithmetic/algebra, and proposing a new order instead. Arnauld's first four "books" (i.e. chapters) developed the foundations of operations with quantities: "La quantité ou grandeur en general", presenting hence an algebra (the commutativity of multiplication can be found postulated here the first time). The application of this general theory of quantities to geometry was given in the following chapters. It is worth noting for this early stage of modern textbook production that all editions of Arnauld's book appeared anonymously.

The subsequent French textbook adhering to Arnauld's model represents the first known case of a debate between mathematicians about the admissibility of negative numbers, thus providing an impressing example of the ongoing methodological reflection in textbooks on the elements of mathematics. Its author, Jean Prestet (1648–1691), was a member of the Order of Oratorians – an Order, although not agreeing with Jansenism, that was also opposed to the Jesuits and was particularly active to promote mathematics and the sciences, in particular, in colleges they ran (Costabel 1988, pp. 79 ff.). Prestet published his textbook, in first instance addressed to the Oratorian college students, in 1675, under the even more aspiring and bold title *Élémens des Mathematiques*. Actually, it was devoted exclusively to arithmetic and algebra, and it contained no geometry at all. Its second edition (1689) was renamed *Nouveaux Élémens des Mathématiques*, constituting likewise a bold programmatical attack on geometry and on synthetic method and presenting the rare case of a self-assured proclamation of the superiority of the analytic method and of algebra. According to Prestet, algebra was more general, while geometry was an applied branch of mathematics. This also explains the book's title: as algebra provides the basis for all mathematics, a textbook on algebra is rightfully called the elements of mathematics.

4.3.1 *The Controversy Between Amauld and Prestet on Negative Quantities*

In the first edition of his textbook, Arnauld had maintained that the multiplication of two negative numbers, such as -1×-1, in principle will give a negative value and only accidentally a positive one:

> MINUS by *minus* gives *plus*: that is to say that the multiplication of two terms, both of which have the sign *minus*, gives a product which must have the sign plus. [...] This appears rather strange, and in fact it cannot be imagined that this could happen other than by accident. For of themselves, *minus* multiplied by *minus* can only give *minus* (Arnauld 1667, p. 13).[2]

This assertion went against the evidence and rigour which Arnauld claimed for mathematics. It was criticised by Prestet in a letter to Arnauld. In his textbook, in 1675, Prestet took the contrary position (Fig. 4.5):


NOUVEAUX ELEMENS
DES
MATHEMATIQUES
OU
PRINCIPES GENERAUX
DE
TOUTES LES SCIENCES
Qui ont les grandeurs pour objet.
SECONDE EDITION, PLUS AMPLE ET MIEUX DIGERE'E.
PREMIER VOLUME
Qui comprend la ſcience des Nombres & l'Algébre, ou l'art de comparer toute ſorte de grandeurs par le moyen des chiffres & des lettres.
Et où tout eſt démontré dans un ordre naturel & facile, & les choſes expliquées plus à fond, & pouſſées plus loin que l'on n'a fait juſqu'ici.
Par JEAN PRESTET Prêtre, ci-devant Profeſſeur des Mathématiques dans les Univerſitez d'Angers & de Nantes.

A PARIS,
Chez ANDRE' PRALARD, ruë ſaint Jacques, à l'Occaſion.
M. DC. LXXXIX.
AVEC PRIVILEGE DU ROY.

Fig. 4.5 Cover of the second edition of Prestet's book (1689)

[2] Moins en moins donne plus: c'est- à dire que!a multiplication de deux termes qui ont tous le signe de moins donne un produit qui doit avoir le signe de plus. [...] Cela paraist bien etrange, et en effet il ne faut pas s'imaginer que cela puisse arriver autrement que par accident. Car de soy-meme moins multiplie par moins ne peut donner que moins.

> The nothing or the zero serve us as a middle term, to make comparisons between the quantities, and for assessing their relations (Prestet 1675, p. 3).
>
> The + and the – of equal magnitudes are each mutual takings away (ibid., p. 10).
>
> When the number to be taken away is greater than the number from which one takes it, the difference or the remainder is negative (ibid., p. 17).[3]

Prestet published his correspondence with Arnauld in the second edition of his own textbook. Besides his own arguments, as in his first edition, it contained the answering letter, Arnauld admitted that he was embarrassed by his own assertion and was going to revise that part, in his second edition. Nevertheless, he gave four reasons against the acceptability of negative quantities:

- It is clear that the operation: 5 *toises* - 2 *toises* [*toise* = fathom] is a legitimate one. But what can be: 5 *toises* - 7 *toises*? This is an impossible operation.
- How is the equation $(-5)^2 = (+5)^2$ possible?
- A crucial reason for Arnauld: according to the notion of proportion,
- 1: – 4:: – 5: 20, although seemingly correct, contradicts that notion, since the third term should be bigger than the fourth - in order to correspond to the order of the two first terms.
- What can −10.000 *ecus* mean? Isolated negative quantities cannot exist.[4]

In his answer to these four objections, Prestet argued in favour of legitimating mathematical objects by ideal operations and not necessarily by concrete objects (Prestet 1689, II, pp. 370–371).

While Prestet had not changed significantly his conception in his second edition of 1689 (Schubring 2005, pp. 58 ff.), Arnauld had considerably revised his conceptions in his second edition, in 1683. Subtraction was restricted to give positive results: the subtrahend had to be less than the minuend (Arnauld 1683, p. 7). And he no longer maintained that minus by minus would essentially result in minus. Yet, he added an extensive discussion of the four cases of the rule of signs, closing with the revealing observation that “the multiplication of minus by minus […] does not conform” to the normal “sort of multiplication”, being “a different kind” of multiplication (ibid., p. 19).

[3] Le rien ou le zero nous sert de milieu pour faire les comparaisons des grandeurs, et pour juger de leurs rapports.

Le + et le − des grandeurs egales se sont l’un à l’autre des retranchements mutuels.

Lorsque le nombre à retrancher est plus grand que le nombre duquel on le retranche, La difference ou le reste est negative.

[4] *Ecu* was a French monetary unit.

4.3.2 From the Seventeenth to the Eighteenth Century in France

Two other members of the Oratorian Order continued and extended Prestet's textbook practice: Bernard Lamy (1640–1715) and Charles-René Reyneau (1656–1728). Lamy's textbooks were particularly successful; their numerous editions were used until the second half of the eighteenth century.

Lamy published two textbooks. The first of these volumes concerned algebra, which was understood as the basis of all mathematics: *Traité de la grandeur en général qui comprend l'arithmétique, l'algèbre, l'analyse et les principes de toutes les sciences qui ont la grandeur pour objet* (1680). The second was devoted to geometry: *Les Elémens de géométrie ou de la mésure du corps* (1685). He proposed that the book on quantities should be read first, and the one on geometry only thereafter, since geometry does not provide an adapted beginning of learning mathematics for it appeals to the sense which might be misleading (Bello Chávez 2021, p. 222). One can understand this sequential order of using his two textbooks as the first case of a series of textbooks, as it became more common since the nineteenth century.

Lamy's textbooks are distinguished by their focus upon reflecting the foundations of mathematics, and by their intention to achieve generalisation. And they reveal the rare case of a direct and strong impact of Descartes's work on this process of generalisation. Each of the two textbooks exhibits a voluminous closing chapter *De la Méthode*. The textbooks' sections are basically constructed as a sequence of rules; noteworthy is the respective major section *Les Principales Regles de la Méthode* which appear, agreeing with Pascal's and Arnauld's approaches as eight rules. These chapters show less fundamental intention than Arnauld's and Prestet's reflections on method; rather, they mark the transition to reflections pertaining more to intra-mathematical issues. The methodological parts are rather exposed as regarding arithmetical/algebraic and geometrical subjects and rules for solving problems.

Lamy shared with Arnauld and in particular with Prestet the sharp critique of the *anciens*: the works of the Ancients, he said, not only lacked *netteté* and *clarté*, they were also too long and too complicated – and above all, they were deficient regarding methodological order (Lamy 1692, *Préface*, [12]). This is why he had tried to transform their demonstrations into "general" ones to make them prove several truths at the same time (ibid., [14 f.]). Since the *grandeur en général* constituted the overall subject in his algebra textbook, this book provided the foundation for mathematics as a whole, hence constituting the true "Elements" of mathematics. Euclid had considered, according to Lamy, only one "particular species" of quantities, namely the geometrical. Lamy declared that didactically it was particularly dangerous because such a textbook supported those who were forever in need of pictures and figures for their demonstrations. Imaging, however, was always a considerable cause for errors. His own textbook, against that, did not require "de se representer des corps", thus no figurative images (ibid., [16 f.]).

Lamy had repeatedly revised the respective parts on method in the different editions, substituting, in particular, the basic presentation of *analyse* and *synthèse* as general methods. Jhon Helver Bello Chavez has analysed in his PhD thesis these changes in the seven editions of the chapter on methods in the geometry textbook, revealing significant differences in the conception of method, and in particular regarding the role of diagrams in geometry. In the first edition of 1685, diagrams had a decisive function: The statement of a problem showed that the diagram, as in Descartes's work, is an essential tool in exploring the solution. The diagram leads to the algebraic solution of the geometrical problem, facilitating that Euclidean relations can be revealed in the Cartesian manner and put into equations via algebraic techniques and procedures (Bello Chávez 2021, p. 228).

The second group of editions of the geometry book are those published since the 1710 version, which Lamy himself revised and expanded by adding a part on conic sections. Here, Lamy's approach was no longer based on the synthetic method as understood since Pappos – to assume that the problem is known – instead, he transposed his entire procedure to the analytic method, i.e., to search what one not yet knows. He called this the "method of invention" (ibid., p. 234). In this version there is no longer the emphasis upon the construction of the diagram. Furthermore, the methodical rules are changed; instead of eight, there are now six rules, and they are more technical.

The third version is given by the last edition, in 1758, decades after Lamy's death. Here, again, there are many additions and alterations. The names of the revisers are not given, but it is clear that they were members of the Oratorian Order. In the sixth chapter, on method, the methodical approach is altered again:

> Now the method has a name: analytical and synthetic. It is clarified that its use is to solve problems that present an application of algebra to geometry. In this edition it is recognised that the analysis seeks a principle, a fundamental proposition that determines the truth of the assumption itself, an aspect that is verified by the synthesis (ibid., p. 239).

Although two separate methods are described, Lamy recognises the interaction between these two forms (ibid., p. 239). And the role of the diagrams is again altered. In this edition, diagrams – but only those in the sixth chapter, on method – were printed as copper engravings on separate sheets at the end – and this has also changed their role for the solution of the problem:

> In the geometric construction, although the image is mentioned, located in the annexes, the fact that it is not in the same space, together with the verbal register, indicates that, for the moment, the diagram was not part of the problem's argumentation; it should be possible to follow the construction in the mind, without needing the senses (ibid., p. 243 f.).

The evolution of Lamy's editions reveals an increasing transformation from a geometry textbook to one on the application of algebra to geometry. "The diagrams and geometric constructions that justify the solution of equations were used less and less; clearly, diagrams which help to visualise problems are still used, while a single diagram represents multiple problems" (ibid., p. 247).

Reyneau published also two textbooks – and they can also be understood as elements of a textbook series: *La Science du Calcul des Grandeurs en Général, ou les Élémens des Mathématiques* (vol. 1: 1714; vol. 2 posthumously 1736), and *Analyse demontrée, ou la Méthode de résoudre les problêmes mathématiques, et d'apprendre facilement ces sciences* (vol. 1: 1708; vol. 2: 1708). Both textbooks had just one second edition. The first one is about arithmetic and elementary algebra, while the second one aspired a higher level – it is about algebra and differential and integral calculus, then a completely new branch of mathematics.

Both textbooks reveal novel, generalising approaches to conceiving of negative quantities and numbers (see Schubring 2005, pp. 78 ff.). As particularly noteworthy, one finds in his *Science du Calcul* volume a diagram with apparently the first explicit use of the four quadrants (Fig. 4.6).[5] Reyneau was a much stauncher propagator of the analytic method than Lamy. He resolved equations without any reliance on Geometry (Bello Chávez 2021, p. 253).

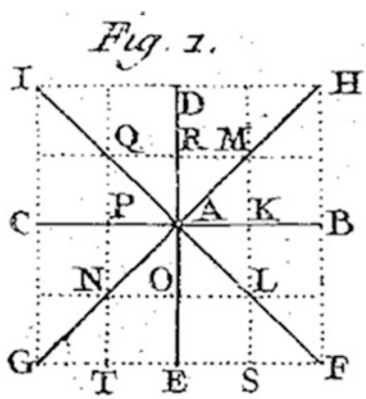


Fig. 4.6 The four quadrants, Reyneau 1736, planche I

In the following decades, many more textbooks of this category were published in France, now by non-clerics, including engineers. I should just mention Dominique-François Rivard (1697–1778), Jacques Ozanam (1640–1717), and Bernard Forest de Bélidor (1697–1761) (see Schubring 2005, pp. 87 ff.). As a textbook of the general category of a particularly revealing level for the eighteenth century, the work by Roger Martin (1741–1811) needs to be highlighted: *Élémens de Mathématiques* (1781). It exposed a novel set of what should constitute "elementary", i.e., teachable mathematics, in what one might call high school level: "à l'usage des écoles de philosophie du Collège Royal de Toulouse", thus for the last years at the secondary school, called those of philosophy, according to the tradition of the Jesuit curriculum. Martin's innovative canon of school mathematics comprised a broad

[5] Reyneau's diagrams were printed as copper engravings, on separate sheets at the end.

range of subjects. The first part, on arithmetic, dealt with integers, fractions, and "irrational and imaginary numbers". The second part, on algebra, exposed general notions, powers and roots, proportions and progressions, logarithms and "analyse", which meant solving equations. The third part taught geometry, including plane trigonometry. A last part, called "elementary notions", contained parts apparently destined to selected students; its first section was on conic sections. The second section, "Principes de calcul Infinitésimal", taught the elements of the differential and integral calculus in a more "elementarised" version than in Reyneau's first attempt to present it as a school subject. It is noteworthy in particular for its conception of limit (Schubring 2005, pp. 225 f.). Martin's book was not just a provincial production; rather, it was well received – even internationally, and had an impact upon Garção Stockler's innovative treatise on the limit (ibid., p. 235).

4.4 The *Sonderweg* in France

At this point, the reader might have noticed a rather privileged role of France in this Pre-Modern period, revealing noteworthy special developments regarding textbook production. This seems to be due largely to the difference of France with regard to other Catholic countries. The Jesuits did not succeed to dominate the educational sector as strongly as in other countries – so that mathematics teaching did not remain as marginal as prescribed in the *Ratio Studiorum*. In fact, the Sorbonne, the most important French university, was not taken over by the Jesuits; and the Sorbonne even did not admit that one of its colleges (as substitutes for the Medieval Arts Faculty) would be taken by them. And in France, there was the strong movement of Calvinism and the activities of the Huguenots, where we have already remarked the strong impact of Ramus's criticism of the methodology of Euclid's *Elements* – with the effect that Euclid never became the standard textbook in France, as it did in other Catholic countries. And later the Jansenism, an intra-Catholic reform movement which was suspected of being aligned to Calvinism, continued this anti-Euclidean movement. The educational policy of the Oratorian Order was acting in an analogous direction. Now, France evidenced even more and particular developments in producing mathematics textbooks, which will be analysed here.

First, I will present an author who published several textbooks which reveal conflicting approaches: Abbé Daniel Deidier (1696–1746). While his school had been a college run by the Oratorians, he studied theology with the Jesuits and became a priest. Later, he became a teacher at a military school, publishing numerous mathematics textbooks, mainly for military engineers. A series of five textbooks is pertinent here:

- *L'Arithmétique des Géométres* (1739)
- *Suite de L'Arithmétique des Géométres, contenant une introduction à l'Algèbre et à l'Analyse* (1739)
- *La Science des géomètres, ou la théorie et la pratique de la géométrie* (1739)

– *La Mesure des Surfaces et des Solides, par l'Arithmetique des Infinis et les Centres de Gravité* (1740)
– *Suite de la Mesure des Surfaces et des Solides: Le Calcul Differentiel et le Calcul Integral, Expliqués et Appliquées a la Geometrie* (1740)

In the second book, in the part on algebra, he had a section on positive and negative quantities, using the conception of opposed quantities. In the part on analysis, i.e., solving equations, he used the synthetic method of presenting procedures for isolated cases, without an explicit conceptual or systematic structure. This part did not require negative solutions, and since he conceived of the coefficients as being positive, he presented four types of equations for solving second degree problems (Schubring 2005, p. 89 f.). Accordingly, geometry constituted for him the superior branch of mathematics.

The two volumes published in 1740 exposed contrary conceptions, even more explicitly. In the first volume, *La Mesure des Surfaces*, he presented the calculus in a synthetic manner, which he explained as based on figures and not on calculating. Deidier criticised the differential and integral calculus of "the Moderns" as being too abstract and metaphysical, leaving the students with obscurities. He declared his conception as one of higher geometry, using John Wallis's arithmetic of the infinite and the method of the centre of gravity, according to Paul Guldin (1577–1643) and André Tacquet (1612–1660). He based himself even on Cavalieri's method of indivisibles (ibid., p. 206).

In the last volume, however, Deidier criticised the synthesis method as limited to details, and praised the enormous progress achieved by Leibniz, "abandoning the grabbling paths of the Ancients". Nevertheless, he tried to combine the analytic with the synthetic method, by addressing the beginners at first with less abstract approaches, but then leading them to more general and rigorous paths (ibid., p. 207).

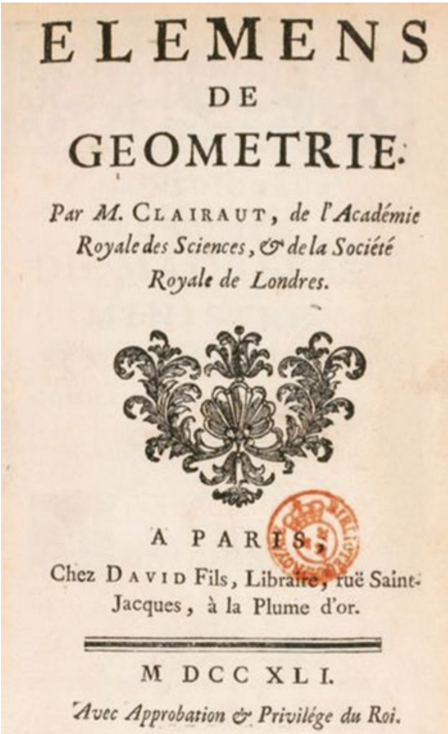
An Unexpected Pattern of the Textbook Triangle

The care for "beginners" instigated a specific didactical approach in the eighteenth century French textbooks: a "gentrified" pedagogy for adults with much leisure time. These textbooks represent an approach which one can call pre-didactic. One has to emphasise the underlying didactical situation in the Pre-Modern Times, where systems of public instruction were still absent: teachers in general suffered from a low social position, being in any case at the beck and call of their pupils' parents. This implied that the teachers were neither in a position to impose a serious learning style, nor to realise effective learning requirements. There was no "contrat didactique" – at least not in the case of upper-class families hiring teachers for private tutoring. This pattern of the textbook triangle revealed not only the already discussed degenerated form (without a teacher), but revealed a rather unexpected variant of this degenerated form: it was the student who dominated the textbook, by determining the conception, the style, and the concept structure of the book (see Fig. 4.7):

There were two protagonists for this textbook approach, aiming more to provide entertainment and amusement than aspiring to teach the essential knowledge: Alexis-Claude Clairaut (1713–1765) and Louis Bertrand (1731–1812).

Fig. 4.7 The second "degenerated" form of the textbook triangle

Clairaut, an eminent French mathematician and physicist, published two textbooks: *Élémens de Géométrie,* in 1741, and *Élémens d 'Algèbre,* in 1746. These books have acquired fame in historiography for allegedly having been the first ones to realise a "genetic approach". This characterisation, however, is misleading; one


ELEMENS
DE
GEOMETRIE.

Par M. CLAIRAUT, *de l'Académie Royale des Sciences, & de la Société Royale de Londres.*

A PARIS,
Chez DAVID Fils, Libraire, ruë Saint-Jacques, à la Plume d'or.

M DCC XLI.

Avec Approbation & Privilége du Roi.

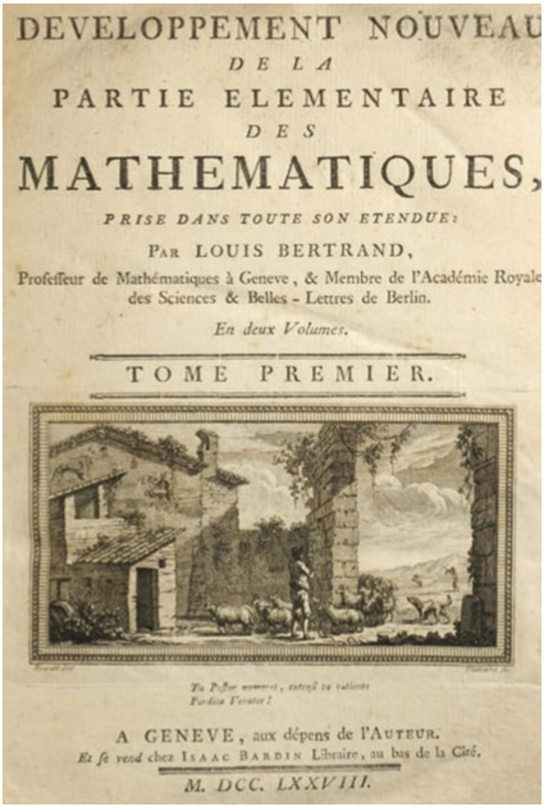
Fig. 4.8 Cover of Clairaut's geometry book of 1741 should better speak of a problem-oriented approach. The geometry textbook intends to develop geometry step by step, always motivated by practical questions of measuring quantities in fields, in the landscape, in farming, and generally in land surveying (Fig. 4.8). For the algebra textbook, the problem-oriented approach was evidently even more difficult to realise.

France could indeed boast about a larger audience for issues of science, in particular among its nobility. These adults would not have liked to be confronted with raised didactical demands for complicated subject matters. Authors like Clairaut avoided presenting those parts of mathematics requiring proper efforts from the reader. His textbooks were not written for teaching in schools; they were written for a Marquise. Therefore, Clairaut did not even mention the problem of the axiom of parallel lines. Likewise, he omitted a discussion of logical procedures, like the reasoning by the absurd. As Glaeser has put it, "Clairaut made fun of the exigencies of mathematics and pretended that the appeal to observation, to common sense and to intuition was sufficient to develop elementary mathematics" (Glaaeser 1983, p. 338). While shunning questions of mathematical rigour, Clairaut claimed that mathematics could only be developed from practical problems. This, however, is a pre-didactical approach since it presupposes that students remain passive while circumventing difficulties inherent to mathematics in order not to scare off beginners.

In the key article on textbooks in the *Encyclopédie* (see next chapter), d'Alembert criticised Clairaut's textbooks for omitting essential proofs and hence for lack of rigour. Besides, he was criticised for providing nothing but a sample of propositions instead of providing a methodical architecture: "forment plûtôt un assemblage qu'un édifice de propositions" (d'Alembert 1755, 497 r.). Remarkably, Clairaut's two books were and still are enthusiastically rediscovered so many times in different countries as apparent achievements of a "royal path" in the teaching of mathematics. Thus, there were reprints and translations in all periods. In France, there was a re-edition in 1920 (and reprints from 1987).

Another example of this "pédagogie mondaine" is the Geneva mathematician Louis Bertrand's (1731–1812) geometry textbook of 1778. Rather than presenting a systematic corpus of geometry, Bertrand pretended to develop its principal propositions by treating those practical problems which might have incited the first inventors in geometry. His presentation of geometry is embedded in a seemingly attractive context, such as questions a hunter asks himself while hunting fallow-deer (see the idyllic atmosphere suggested by the title page of his textbook: with a shepherd and his sheep, Fig. 4.9.).

Although Clairaut's (and Bertrand's) approach does not provide the always and again desired "royal way" to an easy understanding of mathematics, it has decisively moulded the discourse on textbooks for at least 60 years, by launching the key word for textbook-methodology: a textbook should follow "la marche des inventeurs", i.e., the way the inventors took in making their mathematical discoveries. Clearly, the state of mathematics education did not permit to make such a concept operational and a successful one, but the mere launching of this keyword was sufficient to ever and again trigger the imagination of philosophers, educators, and textbook authors, and to appear as a "natural" method for presenting knowledge in an evolving manner.



DEVELOPPEMENT NOUVEAU
DE LA
PARTIE ELEMENTAIRE
DES
MATHEMATIQUES,
PRISE DANS TOUTE SON ETENDUE:
PAR LOUIS BERTRAND,
Profeſſeur de Mathématiques à Geneve, & Membre de l'Académie Royale des Sciences & Belles - Lettres de Berlin.
En deux Volumes.
TOME PREMIER.

A GENEVE, aux dépens de l'AUTEUR.
Et ſe vend chez ISAAC BARDIN Libraire, au bas de la Cité.
M. DCC. LXXVIII.

Fig. 4.9 Cover of Bertrand's 1778 geometry

4.5 New Areas in Mathematics and Their Textbooks: Analytic Geometry and Differential and Integral Calculus

The seventeenth and the eighteenth century also provide instructive examples of how new mathematical developments rapidly develop, allowing to present the new results in textbook format – at first for a scholarly public. The telling examples are analytic geometry and differential and integral calculus.

Descartes published as research his new developments for solving geometric problems by means of algebra with equations, in 1637. It is noteworthy that he published it in French, but it reached international attention when it was translated into Latin, in 1649. The translation was more remarkable because it was not just an isolated work; rather, it was a part of a first cooperative study by mathematicians: dedicated to understand and develop Descartes's new conception. It was organised and coordinated by the Dutch mathematician Frans van Schooten Jr. (1615–1660), a mathematics professor at the engineering school attached to the University of

Leiden who had cooperated closely with Descartes. Van Schooten studied with Dutch students and with French mathematicians those conceptions, and they jointly further developed them. The collaboration led, probably for the first time, to the publication of anthologies with contributions from several mathematicians: the 1649 volume, with the translation contained elaborations. The second edition, published between 1659 and 1661, had two volumes, containing contributions by six authors in addition to Descartes' translation (van Schooten 1659–1661). The contributions by Jan de Witt (1625–1672): *Elementa Curvarum Linearum, Liber Primus,* and *Liber secundus*, which are now available in a bilingual edition (de Witt 2000, 2010), were particularly important for the development of analytical geometry. They can be even regarded as the first textbook on analytic geometry.

Regarding the eighteenth century, there was a greater number of pertinent textbooks; among them, two were well specialised ones: *Introduction à l'analyse des lignes courbes* (1750), by the Swiss mathematician Gabriel Cramer (1704–1752); and *Versuch einer analytischen Abhandlung von den Kegelschnitten* (1759), by Johann Michael Hube (1737–1807), a student of Abraham Kästner in Göttingen. The second volume of Euler's famous textbook *Introductio in analysin infinitorum* (1748) is essentially on analytic geometry. By 1800, one can remark a considerable number of textbooks with 'analytic geometry' in its title; for instance, *Essai de géométrie analytique*, by Jean Baptiste Biot (1802), and *Anleitung zur analytischen Geometrie*, by the Danish Thomas Bugge (1816).

The differential and integral calculus were presented as a textbook, almost immediately after its first research publication. As it is well known, this did not occur for Newtonian calculus since Newton circulated his results only privately among his colleagues. It was Leibniz's publication of a paper in a journal, in 1684, which instigated a first textbook – albeit its imperfect printed form – already in the following year by an Englishman, almost under Newton's eyes: John Craig's (1663–1731) *Methodus figurarum lineis rectis et curvis comprehensarum quadraturas determinandi* (1685), in Leibnizian notations. Most famous and influential was, however, the textbook by the Marquis Guillaume François Antoine de l'Hôpital (1661–1704), who published in 1696, after having been privately introduced to this novel calculus by Johann Bernoulli (1667–1748), *Analyse des infiniment petits*. Remarkable, while published in French, its dissemination was immediate, contrary to Descartes's case. Likewise remarkable is that the following textbook authors were conscious of the process of transforming the research knowledge into teachable or learnable knowledge. The Swiss mathematician Pierre Crousaz (1663–1750), author of the calculus textbook *Commentaire sur l'Analyse des Infiniment Petits* (1721), pointed out that the first generation of calculus textbooks, in particular de l'Hôpital's volume, had been addressed to the *savants* themselves. He understood the later textbooks, including his own, as of a second generation – addressing a much broader public (Crousaz 1721, Préface).

In fact, it is highly revealing to observe how quickly just calculus textbooks were differentiated according to different publics. I have done a comparative research upon these textbooks, since their beginning until the 1820s, regarding four important European countries: France, Germany, England, and Italy. I am reproducing here the two tables of my publication (Schubring 2009a, p. 439 f.) which summarise the results. The names of the authors are there printed in different font styles to indicate the public to which the textbooks were intended (Tables 4.1 and 4.2):

– normal style: for a general (or not identifiable) public;
– *italics*: for university public;
– **bold**: for technical formation;
– underlining: for secondary schools.

Table 4.1 Calculus textbooks published in Great Britain and Italy

Appendix I: Table I

Years	Gr. Britain	Italy
1684	1685 Craig I	
1690		
1700	1704 Hayes; 1706 Ditton	
1710		
	1715 Taylor	
	1718 Craig II	
1720		
1730	1730 Stirling; Stone (L'Hosp. Engl.)	1731 Brunetti
1734	Berkeley	
	1736 *Colson*/Newton; **Muller**; Hodgson; Bayes	
1740	1737 **Th. Simpson**; Smith; 1741 Rowe	
	1742 MacLaurin; 1743 Emerson	1743 **Corradi d'Austria**
		1748 *Agnesi*
1750	1750 **Th. Simpson**	
	1756 *Saunderson*	
	1758 Lyons	1757 *Marzucco*; 1759 **Lagrange**
1760		1761 Martini
		1765-67 *Riccati-Saladini*
1770		1769-71 Caraccioli; 1772 Gaudio
	1777 Holliday	1774-77 *Gherli*; 1779 Antoni
1780		1781 Canovai (Marie ital.)
		1786 Caravelli
1790		
	1795 *Vince*	1793 Franchini
1800	1801 Agnesi Engl.	
1810	1810 Dealtry	1811 **Brunacci**
	1816 Lacroix Élém. Engl.	
1820		
1830		

Table 4.2 Calculus textbooks published in France and Germany

Appendix I: Table II

Years	France	Germany
1684		
1690		
	1696 L'Hospital	
1700		
	1708 *Reyneau*	
1710		1710 *Wolff* German
		1717 *Wolff* Latin
1720		
	1725 Varignon	
	1727 Fontenelle	
1730		
1740	1740 **Deidier**	
		1743 Vallnagel/Wolff
1750		
	1758 *LaCaille*	
1760	1760 d'Alembert Enc.-entry "limite"	1760 *Kästner*; '62 *Segner*
	1769 **Bézout**	1768 *Matsko*; *Makoe*
1770	1770 *Sauri*; *Marie*	1770 **Tempelhoff**
	1774 *Beguin*; 1775 Agnesi French	1775 *Langsdorf/Wolff*
	1777 Cousin I	
1780	1781 *Martin*	1784 **Massenbach**
		1786 *Karsten*
1789	Revolution	
1790		1790 *Pasquich*
	1795 **Prony**; 1796 Cousin II	'92 Spohr; '92-98 Michelsen/Euler
	1797 **Lagrange**; 97-99 Lacroix I;	1799 **Rohde**
1800	an VI ('98) **Bossut**; 1800 **Arbogast**	
	02 **Lacroix** II: Élém.; 04 Desponts	
		'07 *Langsdorf*; '09 *Buzengeiger*
1810	1810 DuBourguet; '11 **Garnier**	'10 **Textor**; '11 J.C. Fischer; *Bohnenberger*
	1813 **Boucharlat**; '14 **Servois**	'13 **Crelle**; '17 *Wrede*; Nürnberger
	1815 **Poinsot**	1818 *T. Mayer*; 1819 *Lehmus*
1820	1821 **Cauchy**	1820 Emmel; *Brandes*; *Schweins*
		1823 *Spehr*; Vestner; Vieth
		'25 **Busse**; **Eytelwein**; Grebel; Unger
		1826 *Siber*; Brosius; J.H. Müller
		'27 *Ettinghausen*; '29 *M. Ohm*
1830		

As the tables show, the publication of calculus textbooks began for a general public, which at first meant scholars. Relatively soon, in particular in Northern Germany, textbooks for a university context arose. For practically the remainder of the eighteenth century, there are textbooks published in a rather parallel manner for a general public, for a university context and for the formation of engineers.

While there were only some isolated cases of books for a school context, their number increased by the end of the century, due to the French Revolution's impact upon public education and due to Lagrange's algebraising approach. The increase of school textbooks in Germany is most notable from the turn to the nineteenth century.

4.6 Summarising the Stratification of Textbook Publics in Pre-modern Times

As we have seen, the invention of the printing press enormously increased the production of textbooks that were not written for a learned public – as one might have thought, due to the traditional focus of mathematics historiography – but for a socially lower ranging public, for practitioners, and merchants, thanks to their publication in the vernacular. For a certain period, along with practically only the editions of Euclid, these *libri d'abbaco* constituted the dominant type of printed mathematics books in various European countries. It was very important, to understand the manners of how mathematics develops, that this practical incentive for disseminating mathematical knowledge instigated at the same time the unfolding and the sound establishment of a new research area besides arithmetic and geometry: namely algebra. The new discipline of algebra was strongly developed at first in Germany, and its international reception made it a particularly well studied area in Britain, from the seventeenth century.

A new pattern of mathematics textbooks emerged only from the early eighteenth century, and this in German Protestant culture. There, the Faculty of Arts of Medieval Times had not been dissolved and partially integrated into the colleges as a secondary school structure as it was the case in Jesuit-dominated Catholic territories; rather, it had not only been maintained but elevated with the new name Faculty of Philosophy, providing for students graduating from secondary schools (Gymnasium) encyclopaedic higher learning, so that they could enter the three traditional professional faculties. And in these Faculties, mathematics was firmly established by chairs for its professors. Textbooks for this student public established the new type of general textbooks. The characteristic one for this pattern became the series published by philosopher and mathematician Christlan Wolff (1679–1754): *Elementa Matheseos*. Actually, he had published this series first in German, in four volumes, in 1710. The Latin version followed, in 1713, first in two volumes, and later, in 1733, extended to five volumes, in a reorganised structure. While the German version often reprinted – also as an abridged version: *Auszug aus den Anfangsgründen* ... – became the standard textbook for those who studied mathematics in Germany in the first half of the eighteenth century, the Latin version became likewise standard in various European countries. In Latin, there was also an abridged version. The titles of the Latin volumes document well the rather broad, encyclopaedic structure of mathematical studies in that period, connecting pure mathematics with a broad range of applications:

> Christlani Wolffii Elementa Matheseos Universae.
>
> Tomus I Qui Commentationem de. Methodo· Mathematico, Arithmeticam, Geometriam, . Trigonometriam,· planam, et analysin tam finitorum, quam infinitorum complectitur. 1713
>
> Tomus II Qui mechanicam cum staticam, hydrostaticam, · aerometriam atque hydraulicam complectitur. 1715.
>
> Tomus III Qui opticam, perspectivam, catoptricam, dioptricam, sphaerica et trigonometriam sphaericam atque astronomiam· tarn sphaericam, quam theoricam complectitur. Halle u. Magdeburg. 1735.
>
> Tomus IV Qui geographiam cum hydrographiam, chronologiam, gnomonicam, pyrotechniam, architecturam · militarem atque civilem complectitur. ·1738.
>
> Tomus V Qui commentationem de praecipuis scriptis mathematicis, commentationem de commentationem de studio mathematico· recte instituendo et indices in tomos quinque matheseos · universae continent. 1741.

In the entry *Élémens des sciences* of the *Encyclopédie*, published by Diderot and d'Alembert (see next chapter), Wolff's textbook series is explicitly praised: "but no one has given a more extensive and in-depth mathematics textbook than Mr. Wolf [sic!]". Yet, d'Alembert added that one has to take this praise as a relative one: "in general, this work does honour its author, although it is not free from faults; but this is the best or the least bad we have so far" (d'Alembert, tome V, 1755, 497) (Tables 4.1 and 4.2).

An analogous endeavour was undertaken by Roger Joseph Boscovich (1711–1787), as mathematics professor at the Jesuit *Collegio Romano* in Rome and one of Clavius's successors. His textbook series title was quite similar to that by Wolff, *Elementa universae matheseos.* Published in Latin, it also addressed a public of scholars. He published three volumes of this series; the fourth one, on calculus, remained unfinished. This was the last realisation to publish an embracing textbook series for an international public. Leonhard Euler, who also published in Latin and hence for an international public, did so for more specialised textbooks.

From the mid-eighteenth century, Wolff as a pioneer of a broad higher learning mathematics textbook was followed by mathematics professors at the two Northern reform universities, at Göttingen and Halle. Abraham Gotthelf Kästner (1719–1800) and Wenceslaus Johann Gustav Karsten (1732–1787) dominated with their textbook series during the second half of the eighteenth century. Kästner published the first edition of his then four-volume series *Anfangsgründe der mathematischen Wissenschaften,* in 1758. Later, he extended the series to eight volumes with many reeditions. Karsten called his series *Lehrbegriff der gesammten Mathematik* (1767–1777). His series titles show that the intention of covering pure and applied mathematics in a quite broad manner continued:

1. Die Rechenkunst und Geometrie. 1767.
2. Weitere Ausfuehrung der Rechenkunst. Die Buchstabenrechnung. Die ebene und sphaerische Trigonometrie nebst weiterer Ausführung der Geometrie. 1768

3. Die statischen Wissenschaften, nebst. den ersten Gruenden der Mechanik. 1769
4. Die. Mechanik der festen Koerper. 1769.·
5. Die Hydraulik. 1770.
6. ·Beschluss der Hydraulik und die Pneumatik. 1771.
7. Die Optik und Perspectiv. 1775.
8. Die Photometrie. 1777.

While it is evident that these textbook series were destined for a university public, this was not usually explicitly emphasised in the books. They maintained a certain general textbook character. This becomes particularly clear by comparing with a new pattern of textbooks established by the mid-eighteenth century: textbooks intended and explicitly stated for a new type of schools and formation, for the then emerging military and technical schools forming officers and engineers in various states. Such schools and proper textbooks were created in England, Italy, and Germany. However, the most revealing case is – again – the French. It was the director of a military school, the *École de Mézières*, founded in 1748, who commissioned Charles-Étienne Camus (1699–1768) to write a textbook series for teaching mathematics at that school and for examinations. Camus published the four-volume series: *Cours de mathématiques* (1749–1752 in its first edition). Étienne Bézout (1730–1783), who was commissioned in 1764 to write a new series, now for schools training navy officers, became more influential. His *Cours de mathématiques à l'usage des Gardes du Pavillon et de la Marine*, for navy officers, appeared in six volumes, starting in 1764 with an arithmetic textbook and finishing with the sixth volume in 1769, on navigation. It proved that in the schools for future artillery officers a somewhat "lightened" version was necessary. Bézout elaborated a proper four-volume series for them, in 1770; the last one was published in 1772. An in-depth analysis of Bézout's textbook work has been published by Liliane Alfonsi (2011), in her scientific biography of Bézout. Her study documented the numerous reeditions during his lifetime and also during and after the French Revolution, even the ones translated into different European languages.

These textbooks for French military schools became the prototype for the new pattern of "schoolbooks" intended for teaching at a definite, specific type of school within an educational system. The emergence of such schoolbooks was prepared during the *Ancien Régime* in France by Camus's and Bézout's textbook production, but it could only be realised as a consequence of the French Revolution, enabling for the first time to establish a public education system with a definite structure. Conceptually, the emergence of the schoolbook pattern was prepared during Enlightenment. Most revealing for this movement is a document in the *Encyclopédie*, the emblematic work providing the fundamentals for overcoming feudalism and establishing a new kind of society. This document is the seminal entry by d'Alembert: *Élémens des sciences*, which will be analysed in the following chapter. It elaborated the concept of “livres élémentaires”, which is only tentatively translated literally as “elementary” books because the usual meaning of “elementary” is “too simple and without particular exigencies”. Its meaning, according to d’Alembert’s notion of elementarisation, will be discussed in Chap. 5.

Chapter 5
Elements: Elementarisation- Structure of the Discipline

I would like to recall the large-scale curricular reforms of the 1960s: their main rationale was to use the "structure of the discipline" to rebuild the school curriculum. Their underlying problem was whether a relation between scientific knowledge and school knowledge exists. These reform efforts were driven by the optimistic view that scientific knowledge could serve as the basis for school knowledge. Paradigmatic for this optimism was Jerome Bruner's claim that every scientific concept can be taught to children in an intellectually honest way.

The activists of this programme were convinced that it was enough to analyse the structure of the discipline, to identify its basic concepts as elements of knowledge, and to reconstruct the school curriculum with that.

It is revealing that an analogous approach was propagated in d'Alembert's contribution to the *Encyclopedie*, in the wake of the profound educational reforms effected during the French Revolution, and of the subsequent establishment of the first school system providing general education.

This approach is neatly circumscribed in the remarkably exhaustive contribution *élémens des sciences* to the famous key work of modernisation, the *Encyclopedie* (tome V, 1755). It is indeed most unusual for a scientist of eminent rank to author a serious reflection on the elementarisation of knowledge and on writing textbooks. D′Alembert's eminent reflections have remained rather unnoticed in the history of science, and since they are quite voluminous and without explicit order, I will try to present their reasoning and structure here.

5.1 D'Alembert's Concept of *Élémens*

In general, "elements" mean the primitive and original parts by which an entity is constituted:

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_5

> On appelle en général *élémens d'un tout,* les parties primitives & originaires dont on peut supposer que ce tout est formé (d'Alembert 1755, p. 491 col. l).[1]

As far as science is concerned, the notion of elements implies that there exists a continuous and logical seriation of all the propositions of a given science into which all elements can be most naturally integrated. Ideally, all propositions can be deduced from just one basic element:

> suppose that this science is entirely treated in a volume, so that we have immediately and before our eyes the propositions, both the general and particular ones, which form the whole of the science, and that these propositions are arranged in the order the most natural & the most rigorous that it is possible: let us then suppose that these propositions form an absolutely continuous sequence, so that each proposition depends uniquely & immediately on the preceding ones, & that it does not suppose any other principles than those that the previous propositions contain; in this case each proposition, as we remarked in the preliminary discourse, will only be the translation of the first one, presented under different aspects; everything would consequently be reduced to this first proposition, which one could regard as the element of the science in question, since this science would be entirely contained therein. (ibid.).[2]

In real life, however, such a common chain enclosing an entire science does not exist - neither for knowledge in general nor for specific branches of knowledge. Or, in d'Alembert's words, we are not even in a position to overlook parts of that chain, and all kinds of order we try to establish will always contain some void:

> But we are far from being able to adopt such a point of view. Far from perceiving the chain which unites all the Sciences, we do not even see in their totality the parts of this chain which constitute each science and to observe in the deduction, there will always necessarily be voids there (ibid., p. 491 col. r).[3]

D'Alembert claims that although many objects escape our attention it is easy to distinguish the general propositions or truths, which serve as basis for the others, implicitly containing them:

---

[1] The pages of the *Encyclopédie* were printed in two columns. The pagination left column or right column is, therefore, here given as either col. l or col. r.

[2] supposons que cette science soit entierement traitée dans un ouvrage, ensorte que l'on ait de suite & sous les yeux les propositions, tant générales que particulieres, qui forment l'ensemble de la science, & que ces propositions soient disposées dans l'ordre le plus naturel & le plus rigoureux qu'il soit possible: supposons ensuite que ces propositions forment une suite absolument continue, ensorte que chaque proposition dépende uniquement & immédiatement des précédentes, & qu'elle ne suppose point d'autres principes que ceux que les précédentes propositions renferment; en ce cas chaque proposition, comme nous l'avons remarqué dans le discours préliminaire, ne sera que la traduction de la premiere, présentée sous différentes faces; tout se réduiroit par conséquent a cette premiere proposition, qu'on pourroit regarder comme l'*élémens* de la science dont il s'agit, puisque cette science y seroit entierement renfermée.

[3] Mais il s'en faut beaucoup que nous puissions nous placer a un tel point de vûe. Bien loin d'appercevoir la chaine qui unit toutes les Sciences, nous ne voyons pas même dans leur totalité les parties de cette chaine qui constituent chaque science et observer dans la déduction, il s'y trouvera toûjours nécessairement des vuides.

> Nevertheless, although there are many objects in this kind of table which escape us, it is easy to distinguish the propositions or general truths which serve as a basis for the others, and in which these are implicitly contained. (ibid.).[4]

Combined, these propositions constitute the elements and can be considered to be the **germ** of the whole:

> These propositions united in a Body, will form, strictly speaking, the elements of this science, since these elements will be like a germ which would be sufficient to unfold to know the objects of science in great detail (ibid.).[5]

D'Alembert explained there are elements that even have a second meaning: those which help us to understand a more detailed and technical presentation of the science in question: "en détail les parties de l'objet plus grossières" (in detail those more brute parts of the object). For d'Alembert's general reflections, however, only elements in the first meaning are relevant.

5.2 How Textbooks Came into Being

In a second part, d'Alembert first gives a rough description of how textbooks came into being. According to him, the evolution of science was slow and discontinuous:

> Most of the sciences were invented little by little: few men of genius, at different intervals of time, discovered one after another a certain number of truths; these have led to the discovery of new ones, until at last the number of known truths has become quite considerable (ibid.).[6]

Eventually, the growth of knowledge created the need for systematic analysis of the discoveries so far achieved. This again led to the first textbooks (traités), which were still quite imperfect:

> The difficulty of adding to the discoveries, of putting in order the discoveries already made; for the character of the human mind uses to first amass as much knowledge as possible and then to think of putting it in order, when it is no longer so easy to amass it. From there were

[4] Néanmoins quoique dans cette espece de tableau il y ait bien des objets qui nous échappent, il'est facile de distinguer les propositions ou vérités générales qui servent de base aux autres, & dans lesquelles celles-ci sont implicitement renfermées.

[5] Ces propositions réunies en un Corps, formeront, à proprement parler, les *élémens* de la science, puisque ces *élémens* seront comme un germe qu'il suffiroit de développer pour connoître les objets de la science fort en détail.

[6] La plupart des Sciences n'ont été inventées que peu-à-peu: quelques hommes de génie, a différens intervalles de tems, ont découvert les uns après les autres un certain nombre de vérités; celles-ci en ont fait découvrir de nouvelles, jusqu'à ce qu'enfin le nombre des vérités connues est devenu assez considérable.

> born the first treatises of all kinds, treated for the most part imperfect & badly structued (ibid., p. 492 col. l).[7]

D'Alembert's major objection to these first textbooks is that the authors were unable to slip into the role of the original inventors. Only these would have been able to treat the subjects to satisfaction:

> Inventors alone could deal satisfactorily with the sciences they had found because in going back over the course of their minds and examining how one proposition had led them to another, only they were in a position to see the connection of the truths and, consequently, to form their chain (ibid.).[8]

The inventors in this early period of scientific development, however, preferred to hide what was truly important: the philosophical principles behind their discoveries. Other inventors, d'Alembert says, were incapable of revealing their methods as they had been guided solely by instinct:

> Finally, there are cases where the inventors themselves could not have reduced their knowledge to a suitable order; it is when, having been guided less by reasoning than by a species of instinct, they are unable to transmit them to others (ibid.).[9]

5.3 Composition of Textbooks

As d'Alembert explains in his next part, it is impossible to write a textbook on the basis of isolated, independent propositions. The main task in developing a methodological order consists in identifying the basic propositions from which all the rest follows. But how to discern these?

> It is therefore evident by all what we have just said, that one should start to establish the elements of a science only when the propositions which constitute it are not isolated and independent of one another, but when one is able to recognise principal propositions of which the others will be consequences. Now how shall we distinguish these principal propositions? (ibid.).[10]

[7] La difficulté d'ajoûter aux découvertes, de mettre en ordre les découvertes déja faites; car le caractere de l'esprit humain est d'amasser d'abord le plus des connoissances qu'il est possible, & de songer ensuite à les mettre en ordre, lorsqu'il n'est plus si facile d'en amasser. De-là sont nés les premiers traités en tout genre; traités pour la plûpart imparfaits & informes.

[8] Les inventeurs seuls pouvoient traiter d'une maniere satisfaisante les sciences qu'ils avoient trouvées, parce qu'en revenant sur la marche de leur esprit, & en examinant de quelle maniere une proposition les avoit conduits à une autre, ils étoient seuls en état de voir la liaison des vérités, & d'en former par conséquent la chaine.

[9] Il est enfin des cas où les inventeurs mêmes n'auroient pû réduire en ordre convenable leurs connoissances; c'est lorsqu'ayant été guidés moins par Je raisonnement que par une espece d'instinct, ils sont hors d'état de pouvoir les transmettre aux autres.

[10] Il est donc évident par tout ce que nous venons de dire, qu'on ne doit entreprendre les *élémens* d'une science que quand les propositions qui la constituent ne seront point chacune isolées & indépendantes l'une de l'autre, mais quand on y pourra remarquer des propositions principales dont les autres seront des conséquences. Or comment distinguera-t-on ces propositions principales?

According to d'Alembert, this can be accomplished by following the chain of propositions back until arriving at one which is not the consequence of a former one. In this perspective, the elements of a science could be reduced to almost nothing: to the real basic set of germ propositions. While d'Alembert admits that it would be a highly impractical "textbook", his idea at the same time reveals his underlying concept of a strictly logical deduction governing the architecture of science:

> Those which borrow something, not only from the preceding proposition but from another original proposition, would seem to need to be excluded for the same reason, since they are implicitly and exactly included in the propositions from which they derive. But by sticking scrupulously to this rule, we would not only reduce the elements to almost nothing but also make their use and application too difficult (ibid., p. 492 col. r).[11]

On the other hand, d'Alembert was realist enough to see that such a deductivist view does not correspond to the practice of science, as isolated propositions must be included, as well, in order to have at least the germ of the truths this science is about.

> Let us not forget to say that it is necessary to insert the isolated propositions in the elements, if there is any which does not hold either as a principle or as a consequence, to any other; for the elements of a science must contain at least the germ of all the truths which are the object of this science (ibid.).[12]

D'Alembert explained that the pivotal issue for the composition of good textbooks is the investigation of the propositions' *metaphysics.* Ideally, this metaphysics can be found by studying the practice of the "inventors"– they have been or should have been guided by the adequate metaphysics. It is seen that "metaphysics", for d'Alembert, is operationalised by a science's most general and philosophical truth. This metaphysics as a guideline should ensure the simplicity and clarity which characterises the elements:

> But what we must above all endeavor to develop well is the metaphysics of propositions. This metaphysics, which has guided or should have guided inventors, is nothing other than the clear and precise exposition of the general and philosophical truths on which the principles of science are founded. The more this metaphysics is simple, easy, and, so to speak, popular, the more precious it is (ibid.).[13]

[11] Celles qui empruntent quelque chose, non-seulement de la proposition précédente, mais d'une autre proposition primitive, sembleroient devoir être exclues par la même raison, puisqu'elles sont implicitement & exactement renfermées dans les propositions dont elles dérivent. Mais en s'attachant scrupuleusement à cette regle, non-seulement on reduiroit les éléments à presque rien, on en rendroit l'usage & l'application trop difficiles.

[12] N'oublions pas de dire qu'il faut insérer dans les *élémens* les propositions isolées, s'il en est quelqu'une qui ne tienne ni comme principe ni comme conséquence, à aucune autre; car les *élémens* d'une science doivent contenir au moins le germe de toutes les vérités qui font l'objet de cette science.

[13] Mais ce qu'il faut sur-tout s'attacher a bien développer, c'est la métaphysique des propositions. Cette métaphysique, qui a guidé ou dû guider les inventeurs, n'est autre chose que l'exposition claire & précise des vérités générales & philosophiques sur lesquelles les principes de la science sont fondes. Plus cette métaphysique est simple, facile, & pour ainsi dire populaire, plus elle est précieuse.

Here is where d'Alembert professes his profound optimism that scientific progress converges with pedagogical progress. Since identical optimism was formulated by Jerome Bruner in the wake of the decisive curricular reforms of the 1960s and 1970s, it seems that periods of profound social reform decisively promote projects of elementarisation and of popularisation of science. Bruner's famous dictum that every scientific concept can be made understandable to children in a recognisable, intellectually honest way (Bruner 1968, p. 44) neatly corresponds to d'Alembert's convictions:

> Everything that is true, especially in the sciences of pure reasoning, always has clear and sensible principles, and consequently can be brought within everyone's reach without any obscurity (ibid.).[14]

We shall continue with some reflections about how to find out the principles of each science, or where to stop this search for the foundations of a given branch. D'Alembert emphasises that this search is not subject to infinite regression and that it is enough to subjectively understand the basic principles, i.e., to stop where the results are principles **for us:**

> I admit that the principles from which we started in this case are perhaps themselves only consequences far removed from the true principles which are unknown to us, and that thus they would perhaps deserve the name "conclusions" rather than "principles." But it is not necessary that these conclusions be principles in themselves, it is enough that they are this for us (ibid., p. 493 col. r).[15]

In the same vein, d'Alembert says that there are primary and secondary truths, but that this differentiation is more dependent on the language than on subject matter.

D'Alembert's key subject, abundantly discussed in this part, regards definitions and their function within good elements. Definitions are essential for mathematics since they are its principles. They use both terms from everyday language and scientific ones. While one might believe that the former are well-known and need not be further reflected on, their widespread use makes a precise reflection of their meaning truly imperative.

Later, d'Alembert discusses what simple ideas ("idées simples") and abstract ideas ("idées primitives que nous acquérons par nos sensations") are for us. His conclusion emphasises that while definitions should be brief, they should also contain the necessary ideas. Definitions are the unfolding of simple ideas comprised in one term:

[14] Tout ce qui est vrai, sur-tout dans les sciences de pur raisonnement, a toûjours des principes clairs & sensibles, & par conséquent peut être mis a la portée de tout le monde sans aucune obscurité.

[15] J'avoue que les principes d'où nous partions en ce cas ne sont peut être eux-mêmes que des conséquences fort éloignées des vrais principes qui nous sont inconnus, & qu'ainsi ils mériteroient peut-être le nom de conclusions plutôt que celui de principes. Mais il n'est pas nécessaire que ces conclusions soient des principes en elles mêmes, il suffit qu'elles en soient pour nous.

> Such are the general rules of a definition; such is the idea we should form of it, and according to which a definition is nothing other than the development of the simple ideas that a word contains (ibid., p. 494 col. r).[16]

With regard to scientific terms, d'Alembert scathingly criticises the (contemporary) wilful and artificial introduction of new ones.

5.4 Textbook Methodology

In this fourth part, d'Alembert discusses how textbooks should be well written. His key notion here, which determined contemporary debate up to the French Revolution, was "the inventors' order", which should be closely adhered to. He added immediately that he did not mean the order they had actually chosen, but rather, the ideal order they should have observed if proceeding methodically, and this optimal textbook order being nothing but the analytic methodology ("methode analytique"). Despite his critique of the original inventors' methodology, d'Alembert took up Clairaut's keyword of following the inventors' way and idealised it, becoming the major methodological criterion. It was d'Alembert who propagated "la marche des inventeurs" as a general methodological device.

In a statement which has become classical, d'Alembert highly praises this key notion of the Enlightenment and of the early stages of the French Revolution:

> In the first place, should we follow, in treating the *elements*, the order followed by the inventors? First of all, it is obvious that it is not a question here of the order that the inventors did ordinarily really followed, and which was without rule and sometimes without object, but of the one that they could have followed in proceeding methodically. It cannot be doubted that this order is generally the most advantageous to follow; because it is more in conformity with the course of the mind that it enlightens while instructing, that it incites to go further, and that it, so to speak, foretells each step one must follow: this is what is otherwise called the analytical method, which proceeds from compound ideas to abstract ideas, which goes back from known consequences to unknown principles, and which by generalising these, manages to discover these; but this method must still combine simplicity and clarity, which are the most essential qualities that the *elements* of a science must have (ibid., p. 495 col. l).[17]

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[16] Telles sont les regles générales d'une définition; telle est l'idée qu'on doit s'en faire, & suivant laquelle une définition n'est autre chose que le développement des idées simples qu'un mot renferme.

[17] En premier lieu, doit-on suivre, en traitant les *élémens*, l'ordre qu'ont suivi les inventeurs? Il est d'abord évident qu'il ne s'agit point ici de l'ordre que les inventeurs ont pour l'ordinaire réellement suivi, & qui etoit sans regle & quelquefois sans objet, mais de celui qu'ils auroient pû suivre en procédant avec méthode. On ne peut douter que cet ordre ne soit en général le plus avantageux a suivre; parce qu'il est Je plus conforme à la marche de l'esprit, qu'il éclaire en instruisant, qu'il met sur la voie pour aller plus loin, & qu'il fait pour ainsi dire pressentir à chaque pas celui qui doit le suivre: c'est ce qu'on appelle autrement la méthode analytique, qui procede des idées composées aux idées abstraites, qui remonte des conséquences connues aux principes inconnus, & qui en généralisant celles-là, parvient à découvrir ceux-ci; mais il faut que cette méthode réunisse encore la simplicité & la clarté, qui sont les qualités les plus essentielles que doivent avoir les *élémens* d'une science.

D'Alembert added that the *méthode analytique* can primarily be applied to sciences whose subject is inside us, while the *méthode synthétique* seemed sometimes more successful in sciences whose subject is outside us.

As a point of particular importance in the composition of textbooks, d'Alembert discussed the relationship between facility and rigour. He postulates that there can be no contradiction between the two:

> Secondly, one asks which of the two qualities should be preferred in the elements, facility or exact rigor. I answer that this question assumes a wrong point; it supposes that exact rigour can exist without ease, while it is the contrary; the more rigorous a deduction, the easier it is to understand: for rigour consists in reducing everything to the simplest principles. Whence it also follows that rigour properly speaking it necessarily entails the most natural and direct method. The more the principles are arranged in the proper order, the more rigorous the deduction will be (ibid., p. 495 col. r).[18]

D'Alembert concludes this part with discussing how these general guidelines are best applied to the particular sciences he grouped into three types: the historical sciences, the liberal and mechanical arts, and the actual sciences ("sciences proprement dites"), which have pure reasoning as their subject.

5.5 How to Use Good *Elements* in Studying

Under this heading, d'Alembert emphasises a new conception of learning, which signifies a definitive break with the traditional mode of orality, requiring a new active participation in learning. The decisive part for successful leaming no longer falls to the teacher; rather, what is now required is the learner's own reasoning and labour:

> It is not with the help of a master that one can fulfill this task, but with much meditation and hard work (ibid., p. 496 col. l).[19]

This rupture becomes even more evident since the ultimate goal is no longer passive reception or correct transcription but to challenge existing knowledge: to really grasp the genius of the inventors in order to be able to master them and to be more creative. Contrary to Clairaut who shirked from requiring intellectual· activity, d'Alembert stressed the labour necessary for such a new active role:

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[18] On demande en second lieu, laquelle des deux qualités doit être préférée dans des *élémens*, de la facilité, ou de la rigueur exacte. Je réponds que cette question suppose une chose fausse; elle suppose que la rigueur exacte puisse exister sans la facilite, & c'est le contraire; plus une déduction est rigoureuse, plus elle est facile à entendre: car la rigueur consiste à réduire tout aux principes les plus simples. D'où il s'ensuit encore que la rigueur proprement dite entraine nécessairement la méthode la plus naturelle & la plus directe. Plus les principes seront disposés dans l'ordre convenable, plus la déduction sera rigoureuse.

[19] Ce n'est point avec le secours d'un maître qu'on peut remplir cet objet, mais avec beaucoup de méditation & de travail.

> To know the *elements* is not only to know what they contain, it is to know their use, their applications, and their consequences; it is to penetrate into the genius of the inventor, it is to put oneself in a position to go further than him, and this is what one does well only by dedication to study and practice: this is why we will only know perfectly what we have learned ourselves (ibid.).[20]

As a consequence, good textbooks, in d'Alembert's opinion, should not strive to be exhaustive, but rather encourage the learner to become active on his own in developing the notions and concepts of the science.

It is quite revealing that d'Alembert also raises the question whether textbooks should be authored individually or collectively, asking if good textbooks can really be the work of one man and emphasising that any textbook will always rely on all-embracing knowledge of the already existing body of knowledge:

> We must now be in a position to judge whether complete elements of the sciences can be the work of a single man: and how could they be, since they suppose a universal and deepened knowledge of all the objects which occupy men? (ibid., p. 496 col. r).[21]

This section also contains an energetic and emphatic request addressed to the leading scientists to get involved in composing the required high-quality textbooks. Again and again, d'Alembert criticised scientists who preferred the glory of augmenting the edifice of science to lightening its entry hall.

> Solely occupied with achieving new progress in their art in order to elevate themselves, if possible above their predecessors or their contemporaries and more jealous of admiration than of public recognition, they think only of discovering & enjoying and prefer the glory of increasing the building to the care of lighting the entrance (ibid.).[22]

He appealed to their sense of responsibility, arguing that their cooperation in the production of textbooks would enlarge the number of people competent in judging science and other affairs. Moreover, this would help to attract more young able people to science:

> A bit more reflection would have shown how harmful this way of thinking is to the progress and glory of the sciences; to their progress, because by making it easier for happy geniuses to study what is known they are put in a position to add to it more and more quickly; to their glory, because by putting them within the reach of a greater number of people we obtain a greater number of enlightened judges. Such is the advantage that would be produced by

[20] Savoir des *élémens*, ce n'est pas seulement connoître ce qu'ils contiennent, c'est en connoître l'usage, les applications, & les conséquences; c'est pénétrer dans le génie de l'inventeur, c'est se mettre en état d'aller plus loin que lui, & voilà ce qu'on ne fait bien qu'à force d'étude & d'exercice: voilà pourquoi on ne saura parfaitement que ce qu'on a appris soi-même.

[21] On doit être en état de juger maintenant si des *élémens* complets des Sciences, peuvent être l'ouvrage d'un homme seul: & comment pourroient-ils l'être, puisqu'ils supposent une connoissance universelle & approfondie de tous les objets qui occupent les hommes?

[22] Uniquement occupés de faire de nouveaux progrès dans l'art, pour s'élever, s'il leur est possible, au-dessus de leurs prédécesseurs ou de leurs contemporains, & plus jaloux de l'admiration que de la reconnoissance publique, ils ne pensent qu'à découvrir & à jouir, & préfèrent la gloire d'augmenter l'édifice au soin d'en éclairer l'entrée.

> good *elements* of the Sciences, *elements* which can only be the work of a very skilful and very experienced hand (ibid.).[23]

In his conclusion, d'Alembert observed that good elementary books would have beneficial effects. A first immediate one would be to prepare young students to pursue the path to new discoveries by themselves, and a second, indirect one would be the moral beneficial in helping people to discern between true and false discoveries (497 Ii) – a pivotal issue in the struggle of Enlightenment against "prejugés".

The entry "élémens des sciences" contains an additional section, telling that after the reflections about the elements of science in general some comments will be made about mathematics and physics textbooks. This section was not written by d'Alembert (who signed his own contributions as "O"), but by an author of mathematical textbooks, signing as "E": Abbé de la Chapelle (1710–1792). The latter criticises Clairaut's textbooks for their lack of rigour while praising German Christian Wolff's mathematics textbooks and – curiously enough – Chapelle's own.

Crucial in this contribution about what constitutes a good textbook is the underlying hotly disputed issue of what should be considered superior: whether the analytic or the synthetic method.

D'Alembert was a bit undecided here. He took a clearer stand, however, in his own contribution on *Analyse* in the *Encyclopedie.* The *Encyclopedie* actually contains two entries on *Analyse,* a general one by d'Alembert himself, and one on *Analyse* in logic by (X), the Abbé Claude Yvon (1714–1791), who entertained good relations with Étienne Bonnot de Condillac (1715–1780), the important French Enlightenment philosopher.

In his own entry, d'Alembert praised the *methode d'analyse*, saying that one of its major applications lay in mathematics where solving equations is often synonymous with Algebra. He declared that this *méthode* had been the major tool for the mathematical discoveries of the past two centuries.

In line with the reasoning first presented by Prestet, d'Alembert emphasised the advantages of Algebra as compared to the tedious procedures of Geometry. The first method permitted abbreviation and easiness while the synthetic one was clumsy and required much more space, thus impeding an easy grasp.

Also in line with Prestet, d'Alembert praised ·the advantages of the "Modernes" as compared to the "Anciens", as the former used the *méthode d'analyse*.

The author of the Encyclopedia's second entry on *analyse* took the risk of completely reformulating the classical definition of the *méhode d'analyse*. It was not, he said, the proper path from the composite to the simple, characterising this as too imprecise to serve as a definition. He sweepingly replaced it by Condillac's view of

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[23] Un peu plus de réflexion eût faire sentir combien cette maniere de penser est nuisible au progrès & à la gloire des Sciences; à leur progrès, parce qu'en facilitant aux génies heureux l'étude de ce qui est connu, on les met en état d'y ajouter davantage & plus promptement; à leur gloire, parce qu'en les mettant à la portée d'un plus grand nombre de personnes, on se procure un plus grand nombre de juges éclairés. Tel est l'avantage que produiroient de bons *élémens* des Sciences, *élémens* qui ne peuvent être l'ouvrage que d'une main fort habile & fort exercée.

the *méthode analytique*, praising the latter as the key to discoveries, superior to the synthetic method and the best method not only for research, but for teaching as well:

> As this definition is not the most exact, we are allowed to substitute it with another one. *Analyse* consists in going back to the origin of our ideas, in developing their generation, and in making different compositions or decompositions of them in order to compare them by all the sides, which can show their relationships. *Analyse* thus defined, it is easy to see that it is the true secret of the discoveries. It has this advantage over *synthèse*, that it never offers only a few ideas at a time, and always in the simplest gradation. It is an enemy of vague principles, and of all that can be contrary to exactness and precision (d'Alembert 1751, p. 401 col. r).[24]

The author criticised Descartes' conception of innate ideas, praising instead Locke and Bacon whose conception was to start from first ideas subsequently developed by sensation and reflection.

He even dared to criticise those geometers who should have known better the advantages of the analytic method but had instead preferred advocating the synthetic method, adding that they had been bound left their mathematical field and tried to deal with other subject matter.

With their propagation of the superiority of the analytic method, the editors of the *Encyclopédie* faced the problem of how to treat the synthetic method in this context. They found an 'ingenious' solution not to publish a separate entry on its behalf, but to reprint an extract from the Dutch philosopher Gravesande's textbook who had authored a relatively neutral presentation (tome 15, 1765, p. 764).

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[24] Comme cette définition n'est pas des plus exactes, on nous permettra d'en substituer une autre. L'analyse consiste a remonter a l' 'origine de nos idées, a en développer la génération & à en faire différentes compositions ou décompositions pour les comparer par tous les côtes qui peuvent en montrer les rapports. L'analyse ainsi définie, il est aise de voir, qu'elle est le vrai secret des découvertes. Elle a cet avantage sur la synthèse, qu'elle n'offre jamais que peu d'idées à la fois, & toujours dans la gradation la plus simple. Elle est ennemie des principes vagues, & de tout ce qui peut être contraire a l'exactitude & a la précision.

Chapter 6
The Period of the French Revolution

What had been prepared by the Enlightenment movement in Western Europe became political reality: first in France, by the Revolution of 1789, and leading to the establishment – as the structure pertinent for us here – of public education systems. More specifically, the concept of introducing *livres élémentaires* became a key pattern for these new educational systems. This did not occur, however, in a uniform way in European countries. Rather, of the different ways two were specially dichotomic, and the analysis of this opposed manner will be made in the next three chapters: the cases of France and of Prussia, which was then one of the numerous German states to become dominant.

6.1 The Excessive Enthusiasm for Livres Élémentaires

D'Alembert's reflection on the elements of science became highly influential for the elaboration of elementary textbooks in the period of the French Revolution. These efforts provided the key words for the later debates and reflections. This dissemination of d'Alembert's ideas was occasioned and intensified by the immense enthusiasm for the analytical method in the early years of the Revolution, due to the reception of Condillac's ideas by the *Idéologues* (Ideologists).

The *Idéologues* became the dominating intellectual group after the fall of Robespierre and the Thermidor period. They put a lot of emphasis on teaching methods, as a quote from their journal *La Décade* shows:

> It was necessary, as much as possible, to make the art of teaching uniform throughout the Republic: the old methods were very vicious; others had to be adapted. The government

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_6

> wanted to fulfill these two objects by establishing “normal schools” (La Décade, 10. Frimaire an III, vol. 3, p. 462)[1]

According to d’Alembert’s challenge to the most important scientists, the Republic called the most eminent *savants* (intellectuals) to elaborate the best teaching methods and schoolbooks:

> They are charged with work which only presumptuous ignorance could regard as easy; they have to find the best “method” of teaching (La Décade, vol. 3, p. 462).[2]

The work of the eminent *savants* was founded on Condillac’s philosophy and his conception of the analytic method:

> This method will no doubt be based on *analyse*. Locke, Helvetius, Condillac have sufficiently demonstrated that it is only by means of “*l’analyse*” that we can enter with confidence into the sanctuary of science (La Décade, vol. 3, p. 462).[3]

Dominique-Joseph Garat (1749–1833), one of the leaders of the *Idéologues*, exulted that the best method had been found:

> There was no longer any need to search for the best method; it was found (Garat 1795, p. 147).[4]

Such methodological convictions were also applied to mathematics: the mathematics of the Greeks was no longer seen as the model; on the contrary, it came to be criticised as a narrow and static knowledge. A notable characteristic expression of the dominating analytical method was formulated by the physicist Jean-Baptiste Biot in 1803:

> The treatises which have come down to our days show us the ancient geometers limited to the simple elements; their genius is confined within this narrow circle from which it cannot escape. If we look for the cause that keeps such strong heads on such details, we soon see that it is the method. The synthesis, which they make use of, proceeds from truths which do not have with each other an equally intimate connection, it is only by a kind of tact that one guesses which one leads to the goal; one can only hope to achieve it if this goal is very close: the progress of science, by this method, is therefore slow and difficult (Biot 1803, p. 23).[5]

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[1] Il fallait, autant que possible, rendre l’art d’enseigner uniforme dans toute la République: les méthodes anciennes étaient bien vicieuses; il fallait en adapter d’outres. Le gouvernement a voulu remplir ces deux objects par l’établissement des ‘écoles normales’.

[2] Ils sont chargés d’un travail que l’ignorance présumtueuse pourrait seule regarder comme facile; ils doivent trouver la meilleure ‘méthode’ d’enseignement

[3] Cette méthode sera sans doute fondée sur l’analyse. Locke, Helvetius, Condillac ont suffisament démontré que c’est uniquement par le moyen de ‘l’analyse’ que nous pouvons pénetrer avec assurance dans le sanctuaire de la science.

[4] Il n’était plus besoin de chercher la meilleure méthode; elle était trouvée.

[5] Les traités qui sont parvenus jusqu’à nos jours nous montrent les anciens geomètres bornés aux simples élémens; leur génie est comme resserré dans ce cercle étroit dont il ne peut sortir. Si l’on cherche la cause qui retient des têtes aussi fortes sur des pareils détails, on voit bientôt que c’est la méthode. La synthèse, dont ils font usage, procède des verités que n’ont pas les unes avec les autres, une liaision également intime, ce n’est que par une sorte de tact qu’on devine celle qui conduit au but; on ne peut même espérer d’y parvenir que si ce but est trés-rapproché: la marche des sciences, par cette mèthode, est donc lente et difficile.

The dominance of the analytical method meant that it should be practiced almost exclusively. Legendre, for example, also applied the analytical method which, because of his own opinion about it, had a curious impact on the structure of his famous textbook on geometry.

In the preface to his "Éléments de Géométrie", the first genuine "livre élémentaire" in mathematics, written in 1794 for the revolutionary *concours* (contest) for *livres élémentaires* (see below), Legendre expressed his own discomfort with the prevailing practice of requiring just this method, the analytical one. He emphasised that the simplest method depends on the respective subject:

> It would be childish to always employ a laborious method when one can substitute it with a much simpler and just as sure one (Legendre 1794, p. viii).[6]

If he had added trigonometry to his schoolbook, he would have used "la méthode ordinaire qu'on appelle méthode synthétique" (the common method called the synthetic method) (ibid.) for his proofs. However, in the first edition, in 1794, Legendre acted according to the dominant view and abandoned trigonometry, including it only in later editions, after the end of the revolutionary period.

It was within this context of dominance of the analytical method that the conception of *livres élémentaires* became a major issue in French educational policy. Indeed, the ideas developed in d'Alembert's entry "élémens des sciences" soon spread and were accepted as a guideline for reform.

Actually, this strong reception soon was evidenced. After the Jesuits were expelled from France in 1762, like they were before from Portugal, prior to the general dissolution of the Order in 1772, intense efforts were made for educational reforms. One of the most influential reform plans was that proposed by Louis-René de Caradeuc de La Chalotais (1701–1785), a French jurist, in 1763. In his opinion, good schoolbooks would be a substitute for systematic teacher training, as using such books would train them for the job:

> These books would be the best instruction that the teachers could give and would take the place of any other method. These well-made books would dispense with trained teachers; it would no longer be necessary to argue about their quality, whether they were priests, or married, or celibate. All will be good, provided they have morals, religion, and know how to read well; they would soon train themselves by teaching children. So, it would just be about having books, and I say that's the easiest thing right now (quoted from Schubring 1988b, p. 159).[7]

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[6] Il seroit pueril d'employer toujours une méthode laborieuse tandis qu'on peut lui substituer une beaucoup plus simple et aussi sûre.

[7] Ces livres seraient la meilleure instruction que les maîtres pussent donner, et tiendraient lieu de toute autre méthode... Ces livres bien faits, dispenseraient de maîtres formés; il ne serait plus nécessaire de disputer sur leur qualité, s'ils seraient prêtres, ou mariés, ou célibataires. Tous seront bons, pourvu qu'ils eussent mœurs, de la religion, et qu'ils sussent bien lire; ils se formeraient bientôt eux-mêmes en formant les enfants. Il ne s'agirait donc que d'avoir des livres, et je dis que c'est la chose la plus aisée présentement.

This is a revealing position regarding the textbook triangle, evidencing an ongoing conflict between teacher and textbook or, concerning the student and the classroom, between orality and printed material. The optimistic stance of La Chalotais that one could substitute teacher training with the use of a genuine *livre élémentaire* is characteristic to the situation that no state so far had established forms and institutions for teacher education. Furthermore, La Chalotais expresses that he was not even able to think of any institutionalised form of teacher education.

This position was not a problem as long as teaching aimed only at the correct transmission of standard texts. Conflicts arose when the functioning of the textbook triangle changed - namely when teachers began to claim to be masters of knowledge themselves, capable of using the text from a higher standpoint, that is, as soon as teachers also became scientifically trained. This was enhanced as a possibility by the Enlightenment with its claim that science should be accessible to the public and form the basis for social life. This radically new dimension of knowledge dissemination freed learning from traditional institutions and thus from the traditional practice of orality as well. Self-study should be possible on the basis of high-quality textbooks.

As a rule, textbooks dominated in countries where teaching and research remained separate, in France, for instance. The opposite extreme was Germany, and particularly Prussia, where teachers, relying on an independence granted to them as "Gelehrte" (scholars), claimed the leading role in the textbook triangle (see Chap. 8). Contrary to that, dominance exercised by textbooks became the model in France.

La Chalotais's principles of educational reform eventually formed the basis for the later reform plans during the French Revolution, supplemented by the complementary notion of guaranteeing uniform instruction by means of a schoolbook adopted for the entire county, and additionally promoted by the intention of a general diffusion of the Enlightenment conceptions, thus enabling the overcoming of the "prejugés" (prejudices).

These prevailing ideas were present and evidenced in all reform proposals and in all debates since the beginning of the French Revolution. One of the first plans, of 1791, by Charles-Maurice de Talleyrand (1754–1838), a highly influential politician in various periods of the French Revolution, emphasised the key role of textbooks in establishing a new school system:

> It is necessary […] that the *livres élémentaires* disseminate universally familiar all the truths (quoted from Schubring 1988b, p. 160).[8]

[8] Il faut [...] que des livres élémentaires rendent universellement familières toutes les verités.

And the famous plan for public education in 1792 by Condorcet,[9] which provided the conceptual framework to which all subsequent decisions had to resort in some way, declared that the writing of *livres élémentaires* was the essential tool for the educational reform, in particular for the training of teachers. In addition, the plan recommended that a further distinction be made between schoolbooks for students and books that serve as guides for teachers:

> Other [books] will be made, which will serve to guide the teachers. These will contain principles on the method of teaching, of forming young people in civic and moral virtues; explanations and developments of the objects contained in the *livres élémentaires* of the schools (quoted from Guillaume 1887, p. 1605).[10]

Condorcet's proposal for introducing specific method books for teachers meant a new and structurally highly important differentiation of textbook types; there were soon the first publications of this type (see below).

6.2 The *Concours* for *livres élémentaires*

After all, after numerous plans had failed, and after the traditional school system collapsed, the first concrete steps were proposed. Quite typically, this was done by means of a *concours*, a public writing competition for *livres élémentaires*! There was the widespread optimism that all the necessary knowledge and qualifications of prospective authors existed, and that the Republic had only to appeal to them in order to promote social welfare.

The *Commission d'instruction publique* (Commission for public instruction) submitted the project of this *concours* to the legislation body. The project was presented to the *Convention* by mathematician Louis François Antoine Arbogast (1759–1803), on December 12, 1792. This project again considered schoolbook production the cornerstone:

> The most effective means for the regeneration of teaching is the composition of *livres élémentaires*. It was of the greatest urgency [...] to hasten the composition of these works (Arbogast 1792, p. 2).[11]

Evidently adhering to d'Alembert's conceptions, Arbogast highlighted the fundamental importance of recruiting the greatest scientists as authors of these books:

> […] there are only the superior men in a science [...], those who have explored all its depths, those who have pushed back its limits, who are capable of composing the elements where

[9] He uses to be called just Condorcet, but his full name was Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743–1794).

[10] Il en sera fait d'autres [livres] qui serviront à guider les instituteurs. Ceux-ci contiendront des principes sur la méthode d'enseigner, de former les jeunes gens aux vertus civiques et morales; des explications et des développements des objets contenus dans les livres élémentaires de l'école.

[11] Le moyen le plus efficace pour la régénération de l'enseignement, c'est la composition des livres élémentaires. Il étoit de la plus grande urgence... de hâter la composition de ces ouvrages.

> there remains nothing to be desired, because they alone can give them that necessary exactness, clearness and intelligibility, and can extract from the entire body of science its fundamental ideas, and the theories which must enter into the *élémens*, serving as an introduction to all the branches known of that science. For perfect élémens, there is nothing too much of the genius of Newton or that of the greatest men who have explained the sciences and the humanities (Arbogast 1792, p. 4).[12]

Political conflicts delayed a parliamentary decision. The next effort in the *Convention* was made on June 13, 1793. This time the proposal passed, but now political instability prevented implementation of the law. The underlying conception had been confirmed, however:

> The elementary works are the columns that must support the building of education (quoted from Schubring 1988b, p. 191).[13]

Eventually and definitively, the first *concours* was decreed by the *Convention,* on January 18, 1794. Within 5 months, schoolbooks had to be presented for ten school subjects; a jury was appointed to evaluate the expected large number of submissions. For mathematics, the members were Lagrange, Monge and Vandermonde. The jury took a year and a half to present the results. The first critical appraisal, however, was given only a few months later, in October 1794. On behalf of the jury, Joseph Lakanal (1762–1845), one of the French politicians very active for education, discussed the notion of *livre élémentaire*: he based it on d'Alembert's conception and refined it to a certain extent in the light of new experiences. In particular, it is noteworthy to remark his distinction between an *abrégé* – a concise version of a voluminous handbook – and an *élémentaire*, that is, a textbook that actually provided the structure of its respective discipline, criticising numerous submitters of manuscripts

> who had generally confused two very different objects, the *élémentaires* ones with *abrégés* ones. To narrow, to straighten a voluminous work, is to shorten it; to present the first *germs* and in a way the *matrix* of a science is to elementarise it: thus, the *abrégé* is precisely the opposite of the *élémentaire* (quoted from Schubring 1988b, p. 161).[14]

D'Alembert's operationalisation of the element as the germ is here not only adopted, but also additionally developed with the term 'matrix'.

[12] "... il n'y a que les hommes supérieurs dans une science... ceux qui en ont sondé toutes les profondeurs, ceux qui en ont reculé les bornes, qui soient capables de faire les élémens où il n'y ait plus rien à desirer; parce qu'eux seuls peuvent leur donner la précision, la clarté et la netteté nécessaires, et extraire de tout l'ensemble de la science des idées fondamentales, et les théories qui doivent entrer dans les élémens, servant d'introduction à toutes les branches connues de la science elle-même. Pour des élémens parfaits, il n'y a rien du trop du génie de Newton ou de celui de plus grands hommes qui aient illustré les sciences et les lettres.

[13] Les colonnes qui doivent supporter l'édifice de l'éducation sont les ouvrages élémentaires.

[14] qui avaient confondus généralement deux objets très différents, des *élémentaires* avec des *abregés*. Resserrer, coarcter un long ouvrage, c'est l'abréger; présenter les *premiers germes* et en quelque sorte la *matrice* d'une science, c'est l'élémenter: ainsi, l'abrégé, c'est précisement l'opposé de l'élémentaire.

Despite the high expectations and the large amount of energy invested in the project, the results were not satisfactory. The jury's final report on all ten subjects considered only seven manuscripts as suitable to serve as *livres élémentaires* and to be printed at the expense of the Republic.

6.3 Results and Effects of the *concours*

The results for mathematics were a disappointment for the protagonists of the project, who were actually forced to change their focus, that is, to abandon a cornerstone of d'Alembert's conception: inventors were no longer considered the primary authors of schoolbooks which had to be chosen:

> Those who have reached the last limits of the area of science do not always have the talent to introduce and guide the inexperienced student step by step (Lakanal 1795, p. 532).[15]

The case of Lacroix's work will show us, in the next chapter, that the predominance of Clairaut's keyword – "suivre la marche des inventeurs" (following the inventors' path) was extinguished in the same vein.

The turnaround was not only due to the meagre results of the competition, but it was also revealed by the creation and short life of the *École Normale* in 1795. As early as 1793, after the collapse of the traditional educational system, it became imperative to take energetic measures. First, such measures had been delayed by the fact that everyone was waiting for the livres élémentaires. This, however, aggravated the situation, as the jury's results came late, in 1795, and their proposals were not approved by the new parliamentary assembly until April 1796!

Since no textbooks had been assigned during that period, projects eventually emerged to establish a genuine form of training teachers. It is very instructive that this training should be organised according to the then well-established model of the revolutionary method ("méthode révolutionnaire"). This method had already been applied before: first at the *École de Mars* to urgently produce the gunpowder and potassium nitrate needed to defend the country against invasion, and then at the famous *École Polytechnique*. The revolutionary institution that was supposed to train teachers for primary schools within just 4 months was called *École Normale* – it became the first higher education institution for teachers. Here, too, the best French scientists were hired as professors.

For a certain time, there was some confusion with the *concours* project: the *Convention* charged the professors of the *École Normale* to compose the new schoolbooks. As several of them were already members of the jury, their conflict of interest rendered the intention unachievable. In the end, the decision was that their lecture notes should constitute schoolbooks. Actually, the *École Normale* established a truly revolutionary method that for the first time realised the unity between

[15] Ceux qui ont atteint les dernières bornes du champ de la science, n'ont pas toujours le talent d'y introduire, et de guider pas à pas, l'élève sans expérience.

research and teaching: lectures by professors, conducted in free speech, were annotated by stenographers employed expressly for this new recording activity. Such notes had to be immediately revised by the professors and were subsequently printed and edited for the students, just a few days after the lecture had been given.

Generally, this new model, which revolutionised the traditional relationship between orality and literality, providing for the first time together with the oral part the originality of researchers' productivity, became common practice: shorthand notes printed as the *Séances des écoles normales* (Sessions of the normal schools) became a masterpiece of elementarisation, and were frequently reprinted.[16]

It has often been discussed how many students attended the *École Normale*. The traditional estimate is 1400, being this large number explained by the demands of the revolutionary method, as student delegates chosen by their districts had to return after 4 months to open primary schools and teach according to their textbooks (the Séances), and according to the methodology they had learned in the lectures. The number of students was proportional to the population of each district. On the other hand, this large number was one of the main reasons for the failure of the stated goal for training teachers in primary schools,[17] as all 1400 students had to attend the lectures together. Although an auditorium for a huge audience had been found in Paris, elementary problems with acoustics made it difficult for many students to follow the lectures. In addition, it was a cold winter, and the Republic faced a financial crisis with its first inflation ("les assignats"). And in addition to these material problems, there were conceptual ones: no one had understood the obvious contradiction between the official objective of training teachers for the **primary** school and the lack of experience of eminent scholars in relation to this charge. In practice, this resulted in lectures in which **secondary** school methods prevailed. The final volume of the series on the history of the *École Normale*, and focusing on its students, is satisfied in commenting that the number of 1.400 students "seems to never have been achieved" (Dhombres 2016, p. 593).

Let us now turn to the result of the competition for mathematics. The texts submitted to the jury had to be unprinted manuscripts. The only manuscript chosen for mathematics as one of seven textbooks is linked to Condorcet's tragic fate. Persecuted by his political enemies, the Jacobins, he lived secretly in Paris, when he was already condemned to death. In that situation, he not only wrote his seminal humanist work *Esquisse d'un tableau historique des progrès de l'esprit humain* (Sketch of a historical picture of the progress of the human spirit) but also occupied himself, after the *concours* was announced, with composing a mathematics schoolbook for the competition. Yet, Condorcet had to flee from his hiding-place before he could finish that manuscript, and his life tragically ended a few days later. Jean-Baptiste Sarret, who lived in the same place and had helped Condorcet out of Paris,

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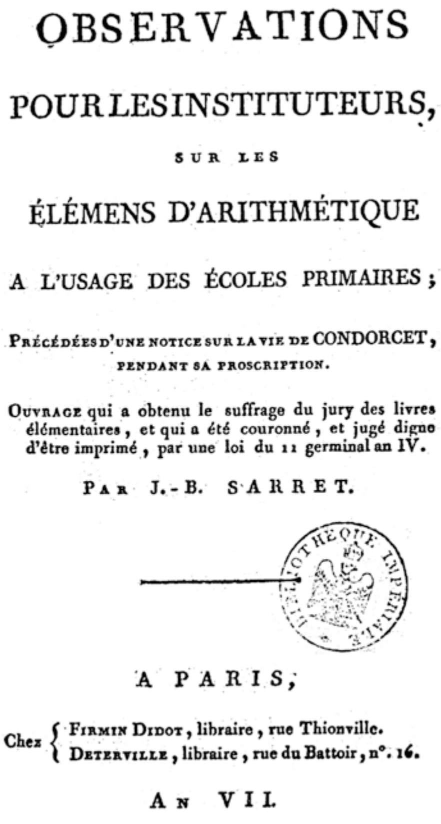
[16] On the occasion of the bicentennial of the French Revolution, a scholarly re-edition of these *Séances* was prepared, which, however, comprising 5 volumes, published from 1992 to 2016.

[17] More exactly, the goal was even more sophisticated: the graduates should, upon return to their districts, train themselves in an analogous way the teachers for the primary schools which then should be created and opened.

obtained a few pages of Condorcet's manuscript, adopted his views, and tried to complete the text where the author was forced to stop.

In the short term, Sarret actually finished writing an arithmetic schoolbook along Condorcet's methodological lines. Both the jury and Parliament later awarded the prize to his manuscript ("*Éléments d'Arithmétique à l'usage des écoles primaires*" – Elements of Arithmetic for Primary School Use), believing the author to be Condorcet. In fact, Sarret had quickly noticed the brilliance of Condorcet's suggestion to publish a methodological guide for the teacher to accompany the students' textbooks (Fig. 6.1). When Sarret came forward as the author of the prize-winning manuscript after a decision by the new *Conseil des Cinq-Cents*, he was accused of plagiarism, in particular by Condorcet's widow. An evaluation by a committee formed by the members of the new French Academy (established to compare two books for the elementary school!) became necessary to clarify that Sarret's text was independent of the handwritten parts that Condorcet had managed to send her from his hiding-place (see Schubring 1988b).



OBSERVATIONS

POUR LES INSTITUTEURS,

SUR LES

ÉLÉMENS D'ARITHMÉTIQUE

A L'USAGE DES ÉCOLES PRIMAIRES;

PRÉCÉDÉES D'UNE NOTICE SUR LA VIE DE CONDORCET, PENDANT SA PROSCRIPTION.

OUVRAGE qui a obtenu le suffrage du jury des livres élémentaires, et qui a été couronné, et jugé digne d'être imprimé, par une loi du 11 germinal an IV.

PAR J.-B. SARRET.

A PARIS,

Chez FIRMIN DIDOT, libraire, rue Thionville.
DETERVILLE, libraire, rue du Battoir, n°. 16.

AN VII.

Fig. 6.1 Cover of the second volume of Sarret's *livre élémentaire*, for the teachers (1799)

Condorcet's own text, later published by his widow, is certainly a revealing document. It is particularly typical of a scholar's approach to the elementarisation of knowledge. Although it is a valuable effort in the elementarisation of scientific concepts, there is no didactic order or progression that makes concessions to children's cognitive abilities. In the second "leçon", Condorcet dealt already with millions and trillions.

One of the outstanding achievements of the text, on the other hand, is the effort for clarity, particularly noticeable in the terminological reforms proposed according to the programme of the analytical method ("*une langue bien faite*" – a well-made language). Condorcet suggested substituting the French words with numbers to express structural analogies: an unambiguously decimal structure should supplant the vestiges of the sexagesimal number system and that with the basis 20 (e.g.: "soixante-dix, quatrevingt-douze"). Tellingly, realising such terminological reforms proved impossible in France, and French children still suffer today from the difficulties of that persisting system.

In 1799, both Sarret's and Condorcet's books were placed on the official list of *livres élémentaires* for French primary schools. Unfortunately, it is not known to what extent or how these two textbooks were used. Condorcet's book was re-edited various times, also in the twentieth century. There were also translations especially in Portuguese.

A few textbooks already in print were also examined by the competition jury; in mathematics, in particular, Legendre's "*Éléments de géométrie*", published in 1794. This book, which won a distinction by the jury, inaugurated a new style – surpassing the former pre-didactic models (see Chap. 4). As Delambre emphasised later, in 1810, in his report to Emperor Napoleon about the progress of mathematics since the Revolution,

> Monsieur Legendre undertook it to revive among us the taste for rigorous demonstrations (Delambre 1810, p. 45).[18]

While the immediate objectives of the programme to elaborate and use *livres élémentaires* were not achieved, and much of the original optimism, in the sense of conceiving elementary books as foundations for the new school system, collapsed, an important question was reinforced in the attempt to achieve the knowledge that the production of textbooks should no longer be aligned with static teaching; but rather, it should be dynamically linked to research. This intention was clearly stated by the philosopher Antoine-Louis-Claude Destutt de Tracy (1754–1836), one of the most important *Idéologues* (Ideologists), the group that dominated educational policy from 1794 onwards, during the first part of the Napoleonic period.

[18] M. Legendre entreprit de faire revivre parmi nous le goût des demonstrations rigoureuses.

During the attempt to realise the social programme of educational reform during the French Revolution, it was clear that the problem of mathematics education is an independent one, and that it creates a crucial transmission mechanism for the development of science. While the *Encyclopédie* had still admitted that science has limits ("les bornes"), the new dominant orientation towards the new role of scientific knowledge, due to its institutionalised teaching, created an impact upon favouring conceptual development. Not only the striving for rigour typical of mathematics, particularly after 1800, but also the more theoretical development of mathematics can be understood in this way. In 1801, based on the experience of the recent years and the *concours*, Tracy emphasised how important the composing of *livres élémentaires* was for the progress of science: A research article, he said, achieves its goal by presenting what is new, but writing a *livre élémentaire* was something quite different. Truths must be arranged in a recognisable order, essential propositions must not be left out, and all parts must be systematically structured to enable the less instructed readers to proceed with ease. The problem can, however, become even more fundamental, since the author when proceeding with his work, may come across gaps in the existing knowledge:

> Often, while exposing a fact, one becomes aware that it requires new observations, and, better examined, it presents itself under an entirely different aspect: at other times, the principles themselves which are to be redone, or to be connect to one another, there are many gaps to fill; in one word, it is not just about exposing the truth, but about discovering it (Destutt de Tracy 1801, p. 4).[19]

Thus, it proved that elementarisation implies an intrinsic relationship between the teaching of a science and the progress of that specific science.

6.4 French State Policy for *livres classiques*

Like French politics in general, educational policy changed dramatically under Napoleon. Using the structures of control already established in the preceding period of the Revolution to fight the *prejugés*, which meant fighting the preconceived ideas and superstitions, the pride and prejudice of local traditionalism, Napoleon extended the powers of the central state, prescribing strictly uniform standards. Committees were established to review and adapt books for use in schools. In fact, the Napoleonic era saw stricter regulations in education, as well as in almost

[19] Souvent, en rendant compte d'un fait, on s'apperçoit qu'il exige de nouvelles observations, et, mieux examiné, il se présente sous un tout autre aspect: d'autres fois, ce sont les principes eux-mêmes qui sont à refaire, ou, pour les lier entre eux, il y a beaucoup de lacunes a remplir; en un mot, il ne s'agit pas seulement d'exposer la verité, mais de la découvrir.

every other field, for the entire French territory. Typical is the 1803 decree obliging schools to use only one approved schoolbook in any subject for a given school level:

> The teacher may not, under any pretext whatsoever, teach other textbooks (Recueil des lois..., tome 2, p. 308).[20]

In a quite paradigmatic manner, the teacher is here explicitly conceived as the "organ" of the textbook – the teacher's task is to convey the knowledge as exposed in the book – he should "speak" as the textbook. The function of the teacher within the textbook triangle is to serve the textbook, in its dominant function.[21]

Abiding by such a policy, there was a radical change in the conception of textbooks and a return to classical values; the programme of *livres élémentaires* was abandoned at the same time when science was achieving enormous discoveries. The 1803 decree stipulated a radical change of values – it was no longer this progress which should be valued, instead, it was the static, the invariable which should provide the values for education:

> The principles of the *belles-lettres* are not subject to the same revolutions as those of the sciences, they are drawn from the imitation of a model which does not change. The teaching of these arts, the essence of which is invariable, has therefore long been subject to rules certainly established, while the sciences, on the contrary, are forced to abandon their old systems every day for the new observations which time or chance bring about. It would be ridiculous today to refer to Ptolemy and Epicurus as authorities for astronomy and physics. But the principles of Aristotle and Horace have not changed at all; eloquence and poetry still follow them (Recueil des lois..., tomo 2, p. 378).[22]

As early as 1799, such a change was remarkable in mathematics when Lagrange recommended Bézout's books as fundamental. This instructive case was analysed by Jean Dhombres and Dominique Julia in an article on the deliberations of the *Conseil d'instruction Publique* (Council of Public Instruction) from March to May, 1799, concerning the selection of textbooks for the *écoles centrales*. The committee's task had been to search for uniformity teaching standards in France ("uniformité des principes") and to achieve this task by choosing the best textbook for each discipline, the books now being renamed "ouvrages classiques". In particular, the

[20] Le professeur ne pourra, sous quelque prétexte que ce soit, enseigner autres ouvrages.

[21] The history of textbook regulations and policy in France has been studied and documented by Choppin, 1980, 1982. The "Emanuelle" database, developed by the *Service d'Histoire d'Éducation* in Paris, published textbook directories for the various school subjects from the Revolution to the present day.

[22] Les principes de belles-lettres ne sont pas sujets aux mêmes révolutions que ceux des sciences: ils sont puisés dans l'imitation d'un modèle qui ne change point. L'enseignement de ces arts, dont l'essence est invariable, a donc pu dès longtemps être soumis à des règles certaines, tandis que les sciences, au contraire, sont forcées d'abandoner tous les jours leurs anciens systèmes pour les observations nouvelles qu'amène le temps ou le hasard. Il serait ridicule aujourd'hui de citer à l'astronomie et à la physique l'autorité de Ptolémée et d'Epicure: mais les principes d'Aristote et d'Horace n'ont point changé; l'éloquence et la poésie les suivent encore.

committee was charged with deciding on the students' books, the volumes to be placed "avec confidence entre les mains des élèves" (with confidence into the students' hands) (Dhombres and Julia 1993, p. 9).

Despite this change, the *Conseil* clearly expressed its continued preference for the analytical method, as it was

> the only one that the Council wants and desires to approve because it is the only one that leads the disciple from the object to the sensation and from the sensation to the ideas and the judgment, makes him arrive at the rules, at the general maxims that he understands well then because they are the result of his own observations (Dhombres and Julia 1993, p. 11).[23]

The Council even emphasised its suitability for general education, since the method presented the subject matter

> by an extreme clarity united with great precision [...] [resulting in bringing] the most abstract things within the reach of minds that are less exercised and capable of a secure application (Dhombres and Julia 1993, p. 11).[24]

Lagrange was the only mathematician on that Council. He proposed the Bézout volumes as the best textbooks to be used by students. For the final vote, Lagrange extended his recommendation. While Bézout was still at the top of the list, Lacroix's textbooks were included – a fact not observed or commented by Dhombres and Julia. This change sheds new light on the rivalry between Legendre and Lacroix (see next chapter). Lagrange essentially gave four reasons for choosing Bézout:

1. Lagrange thought that this textbook made "general education" in mathematics possible, as it integrated all the mathematics necessary for professional use in the navy, artillery, and fortifications (ibid., p. 12). Of course, a general applicability of mathematics even at that time extended far beyond the mere propaedeutics of military engineering and training, and the *écoles centrales* prepared for more than just studies at the *École Polytechnique*.
2. His second reason was style. Lagrange stressed the coherence of text and method in the various volumes of the *Cours* de Bézout, and in particular their structural homogeneity. Lagrange even praised its list of contents:

[23] la seule que le Conseil veuille et désire approuver parce que c'est la seule qui conduise le disciple de l'objet à la sensation et de la sensation aux idées et au jugement, le fait arriver aux règles, aux maximes générales qu'il comprend bien alors parce qu'elles sont le résultat de ses propres observations

[24] par une extrême clarté unie avec beaucoup de précision [...] les choses les plus abstraites à la portée des esprits soit peu exercés et capables d'une certaine application.

It is made in a consistent manner: arithmetic, geometry, algebra, a treatise on mechanics, everything is linked and forms a single body, and each volume ends with a table of contents very well made, being a great resource for students.[25]

Besides this praise, Lagrange underlined the "clarté", clearness, of the text and the exactness of the rules, since Bézout's *Cours* was composed

of purity and elegance, two sorts of advantages which d'Alembert was the first to bring to the abstract sciences and which deserve to be doubly appreciated in an elementary work, especially when, as in this one, they do not harm at all the extreme clarity of the sentences nor the precision of the doctrine.[26]

3. Apparently, economic reasons also prevailed: Bézout's work was no longer private property! As the author had been dead for over 10 years and could not claim copyright, the price could be kept low:

His work no longer has an owner [...] and this would be a reason for the edition which would be made of it to be delivered at a very moderate price to the students of the mathematics classes (Dhombres & Julia 1993, p. 13).[27]

4. Lagrange repeatedly highlighted his fourth reason: Bézout's volumes were a "Cours complet", meaning that the collection contained the complete material of general school mathematics. Dhombres and Julia mention a critical debate in the committee over Lagrange's proposal, sparked by his observation that Bézout was not a first-rate mathematician (ibid., p. 14).

In fact, the controversy was not over Bézout's mathematical merits; there had been a prolonged and reluctant debate within the committee and the ministerial cabinet on selecting just one particular textbook and giving it the legitimacy of official approval. In contrast to the procedure in relation to other school disciplines, the choice of the mathematics textbook and Lagrange's proposal were discussed in several committee meetings. Lagrange presented his first proposal on 8 Germinal of the year VII (28 March 1799), not only recommending Bézout's books as the standard for mathematics, but also as the exclusive standard! The committee deferred its deliberations to the next meeting. Then, on 18 Germinal (April 6), it agreed with Lagrange's proposal, adding that it regarded this preference for an author as a specific exception for mathematics rather than as a model for other disciplines. The decision, the committee commented, was simply to say that Bézout's book had been chosen as the standard textbook for beginners in mathematics to avoid "la confusion des méthodes et la disparité des exemples qu'ils pourraient trouver dans les autres

[25] Il est fait d'un seul jet: arithmétique, géométrie, algèbre, traité de mecanique, tout y est lié et forme un seul corps, et chaque volume est terminé par une table des matières très bien faite et d'une grande ressource pour les étudiants.

[26] avec pureté et élégance, deux sortes d'avantages que d'Alembert fut le premier à porter dans les sciences abstraites et qui méritent d'être doublement appréciées dans un ouvrage élémentaire, surtout lorsque, comme dans celui-ci, ils ne nuisent en rien à l'extrême clarté de la phrase ni à la précision du précepte.

[27] son ouvrage n'a plus de propriétaire... et ce serait une raison pour que l'édition qui en serait faite fût livrée à un prix trés moderé aux élèves des classes de mathématiques.

ouvrages" – the confusion of methods and the disparity of examples that could be found in other works (Dhombres and Julia 1993, p. 14).

The Council had prepared its report for the ministerial cabinet in the form of a draft decree. Apparently, the cabinet was unwilling to recommend a single standard textbook, and the Council had to deliberate again on the mathematics case, asking Lagrange to submit another report and proposal for the meeting on 18 Floréal year VII (May 8, 1799) – this time no longer restricted to the presentation of just one textbook, but providing a long list, which was somewhat encyclopaedic or even historical and did not contain any consideration of didactic aspects. Lagrange included not only Newton's *Arithmetica Universalis* but also Euclid's *Elements*. Furthermore, his report was somewhat contradictory, as it was not restricted to maintaining the earlier logic of recommending complete, "whole" collections, and also listed separate books for school mathematics subdisciplines, such as geometry, algebra, trigonometry, and also included "school subjects", such as descriptive geometry and differential calculus. Although Lagrange still insisted on his strategy of favouring one textbook as the official standard (Bézout), he now mentioned several other "complete" ones (Lacaille, Bossut), adding the *Cours* de Lacroix with the caveat that it was not yet complete, but would be soon, and putting this *Cours* in last place due to the gaps it presented at that time.

In yet another meeting, held on 28 Floréal (May 18), the Council adopted this somewhat unsystematic historical-encyclopaedic selection, but the report accompanying the proposed decree said how uncomfortable most of its members still felt in giving precedence only to one author; also explained that the order in which the various books were listed was not linked to their merit ("ne tient point au mérite"), but rather to a placement according to their historical order, degree of difficulty, or connections to textbooks for related subjects.[28]

We can thus see that educational policy in this more advanced period of the Revolution did consider the teacher to be responsible and capable of choosing his preferred textbook for himself when suggestions were given. Against this position, Lagrange advocated a centralising policy that recommended standards and uniformity. This crucial conflict escaped the attention of Dhombres, who also did not reflect upon whether or not Lagrange's judgment was justified. In the same vein, he omitted to compare the *Cours* de Bézout with previous publications within the programme of *livres élémentaires*.

In his notes for the reprint of the history of the *École Polytechnique* de Fourcy, Dhombres had become aware of the contradiction between the algebraic approach and that of Bézout:

> Bézout's *Cours* did not have a good press at the School during the first years; it lacked an algebraic spirit (Dhombres 1987, p. 101).[29]

[28] Archives Nationales, F[17], n. 1011 A, Protocol of the session of 18 Floréal, year VII, and of 28 Floréal, and attached reports addressed to the ministerial office.

[29] Le cours de Bézout n'avait pas bonne presse à l'École pendant les premières années: il manquait d'esprit algébrique.

Lamandé provided some elements of a critical analysis of Bézout's *Cours*. As he pointed out, the first two volumes in Bézout's collection – arithmetic and geometry – contain almost no algebraic symbols or notation and are almost exclusively verbal. Bézout, who was sincere about preferring the synthetic to the analytic method, was enormously successful with his series, also long-time after his death, and even after his series were no longer essential for examination. In the year VII (1799), when the ministerial cabinet sent a series of questionnaires to the *écoles centrales* with the aim of obtaining information about practical teaching in these revolutionary schools, 50 out of 69 schools reported that they were teaching mathematics according to Bézout. In his reply, Arbogast regretted having been forced to use Bézout because it was already in the hands of his students. And he added:

> We have not concealed that there are other *livres élémentaires* which deserve preference by the choice of subjects, the analytical order which reigns there and the rigour with which they demonstrate (quoted from Lamandé 1990, p. 31).[30]

Lamandé portrays Arbogast as a representative of scientificity, but there are dimensions beyond the contrast between rigour and "facilité" (facility) that Arbogast criticised in Bézout's books. In particular, there is the didactical problem of knowing whether the learner will be instigated or not, by a strategy of "applanir les difficultés" (planishing the difficulties), to arrive at a better understanding of the notions and concepts of mathematics. Bézout's geometry books provide numerous examples of cases where the learner is left without essential elements of various concepts, getting only superficial ideas from them. His style was certainly "elegant" and easy to read, but the content was somewhat vague. This is illustrated by Bézout's exposition of the relationship between curves and polygons: after having given some examples, Bézout states as a general conclusion that "on peut regarder la circonférence du cercle comme un polygone régulier d'une infinité de cotés" (one can consider the circumference of the circle as a regular polygon with an infinity of sides) (Bézout 1803, p. 71). There is no discussion or even mention of the fact that inscribed and circumscribed polygons are needed to justify such a claim.

The fact that Bézout avoided algebraic notation and symbolic writing is by no means only related to arithmetic and geometry, it extends to algebra as well. A typical example is Bézout's presentation of the solution of second degree equations. In contrast to all textbooks of his time, it is entirely verbal, almost without any symbolism, requiring nine lines in the original:

> Take half of the known quantity which multiplies x in the second term, square this half, and add this square to each side of the equation, which will not change the equality. The first member will then be a perfect square. Extract the square root of each member, and precede

[30] on ne s'est pas dissimulé qu'il existe d'autres livres élémentaires qui méritent la préférence par le choix des matières, l'ordre analytique qui y règne et la rigueur avec laquelle on y démontre.

> that of the second member with the double sign ±; the equation will be reduced to the first degree (Bézout 1781, p. 129).[31]

It can therefore be inferred from Lagrange's preference for Bézout that eminent scientists not only may not be the best schoolbook authors, but may also not yet be the best experts in evaluating textbooks.

[31] Prenez la moitié de la quantité connue qui multiplie x dans le second terme: élévez cette moitié au quarré, et ajoutez ce quarré à chaque membre de l'équation, ce qui ne changera rien à l'égalité. Le premier membre sera alors un quarré parfait. Tirez la racine quarré de chaque membre, et faites précéder celle du second membre, du double signe ±; l'équation sera réduite au premier degré.

Chapter 7
Lacroix as an Entrepreneur: His Struggle for the Textbook Market

7.1 The Monopoly of Lacroix's *Cours*

In the previous chapter, we saw that schoolbook policy underwent a radical change towards the end of the *Directoire* period, and that an even more dramatic modification occurred with the onset of Napoleon's rule. The idea that *livres élémentaires*, in alignment with the respective structure of the discipline should elementarise science even in its most advanced status, was abandoned; traditional values and epistemologies were resurrected, and mathematics and science teaching became less important.

Curiously enough, Lagrange himself, not only a representative of the modernisation of mathematics but also, after 1794, an ardent activist of the elementarisation projects, proposed, in 1799, that a single textbook-series was recommended, suggesting the volumes written by Bézout, who was a representative of the historical period just overcome.

After all, the *Conseil d'instruction publique*, responsible for deciding about schoolbooks, decided to offer some choice to mathematics teachers at the *Écoles Centrales*, including, in particular, Lacroix's textbooks, despite the fact that until that moment he had not been able to present a "complete" course, i.e., a textbook-series for all grades. The main titles on the committee's list were[1]:

<u>Schoolbooks admitted for use in French secondary schools.</u>[2]

<u>1799</u> *Écoles Centrales* (optional)

Bézout	Cours de Mathématiques
Lacaille	Cours de Mathématiques
Lacroix	Cours de Mathématiques

[1] Archives Nationales, F^{17}, n. 1011 A.

[2] The books are listed in the document in this abbreviated and shortened form.

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_7

Legendre	Éléments de Géométrie
Clairaut	Éléments de Géométrie
Mauduit	Géométrie descriptive
Newton	Arithmétique universelle

While for the *Écoles Centrales* there had been no officially decreed curriculum, the *Lycées* that replaced them in the Napoleonic era were organised according to rigid principles, and instruction had to adhere to a clearly hierarchical curricular order. As already mentioned, from 1803 onwards, only one textbook was allowed to be used by teachers and students for each level. Within this authoritarian and centralising structure, Lacroix's textbooks had almost a monopoly, with the exception of mechanics, a subject on which he did not publish. Here is the list of this curricular definition:

<u>1803</u> *Lycées* (obrigatório)[3]

Lacroix	Traité élémentaire d'arithmétique
Lacroix	Éléments de Géométrie
Lacroix	Traité élémentaire de trigonométrie et de l'application de l'Algèbre à la géométrie
Lacroix	Compléments des éléments d'algèbre
Francoeur	Traité élémentaire de mécanique

The lists of schoolbooks recommended up to the Restoration period does not always indicate whether these titles were merely added to earlier lists or their items replaced other books. They sometimes just give the authors' names, omitting the book titles. Thus, it can be assumed that later a certain freedom of choice was returned to the teachers. The following list is from 1809:

<u>1809</u> *Lycées*

Bézout	Traités élémentaires d'arithmétique et d'algèbre
Bossut	id.
Marie	id.
Lacroix	id.
Lacroix	Éléments de géométrie
Legendre	Éléments de géométrie

The last list from the Napoleonic period provides only two titles, suggesting it was a supplement to the former list, expressing that choices were possible:

<u>1813</u> *Lycées*

Euler	Éléments d'algèbre
Lacroix	Complément des Éléments d'algèbre

[3] Belhoste 1995, 78–81.

During the Restoration period, the first list of schoolbooks indicated for mathematics in the *collèges*, the again renamed secondary schools, gave two new series, thus competing with the Lacroix one:

1821 *Collèges royaux*

Bourdon	Arithmétique; Algèbre
Reynaud	Arithmétique; Algèbre; Application de l'algèbre à la géométrie

What sort of person was this Lacroix who seems to have aspired to be the exclusive author of mathematics schoolbooks, achieving for a time almost a monopoly in French secondary schools? An impression of the wide reach of his textbook oeuvre and its success can be obtained from his own catalogue of his so-called "Cours Complet", reprinted in one of his books in 1819:

> COURS COMPLET DE MATHÉMATIQUES À L'USAGE DE L'ÉCOLE CENTRALE DES QUATRE-NATIONS; OUVRAGE ADOPTÉ PAR LE GOUVERNEMENT POUR LES LYCÉES, ÉCOLES SÉCONDAIRES, COLLÈGES, ETC., PAR S.F. LACROIX, MEMBRE DE L'INSTITUT ET DE LA LÉGION-D'HONNEUR, PROFESSEUR AU COLLÈGE ROYAL DE FRANCE, ETC., 9 VOL. IN-8.
> Prix pour Paris 38 fr. 50 c.

Chaque volume se vend séparément, à savoir:

Traité élémentaire d'Arithmétique, 14e édition, 1818	2 fr.
Élémens d'Algèbre, 12e édition, 1818	4 fr.
Élémens de Géométrie, 11e édition, 1819	4 fr.
Traité élémentaire de Trigonométrie rectiligne et sphérique, et d'Application de l'Algèbre à la Géométrie, 6e édition, 1813	4 fr.
Complément des Élémens d'Algèbre, 4e édition, 1817	4 fr.
Complément des Élémens de Géométrie, Élémens de Géométrie descriptive, 4e édition, 1812	3 fr.
Traité élémentaire de Calcul différentiel et de Calcul intégral, 2e édition, 1806	7 fr. 50 c.
Essais sur l'Enseignement en général, et sur celui des Mathématiques en particulier, ou Manière d'étudier et d'enseigner les Mathématiques, 1 vol. In- 8., 2e édition, 1816	5 fr.
Traité élémentaire de Calcul des Probabilités, in- 8, 1816	5 fr.
Traité de Calcul différentiel et de Calcul intégral, 2e édition, revue et considérablement augmentée, 3 gros vo. In-4., avec planches.	Prix pour Paris, 66 fr.

As this list is an impressive evidence of the success of Lacroix's textbooks, and in particular those intended for school subjects, is attested by the enormous number of re-editions. In addition, Lacroix became the most frequently translated French author. In fact, his books were used throughout Europe, as well as in the Americas (see Oliveira and Schubring 2021).

7.2 Lacroix: A Textbook Author at the Level of the Inventors

Sylvestre-François Lacroix (1765–1843), protected by Condorcet at the beginning of his career, was a mathematics teacher in military schools during the *Ancien Régime*. His rise occurred after the Revolution, and he accumulated several important positions.

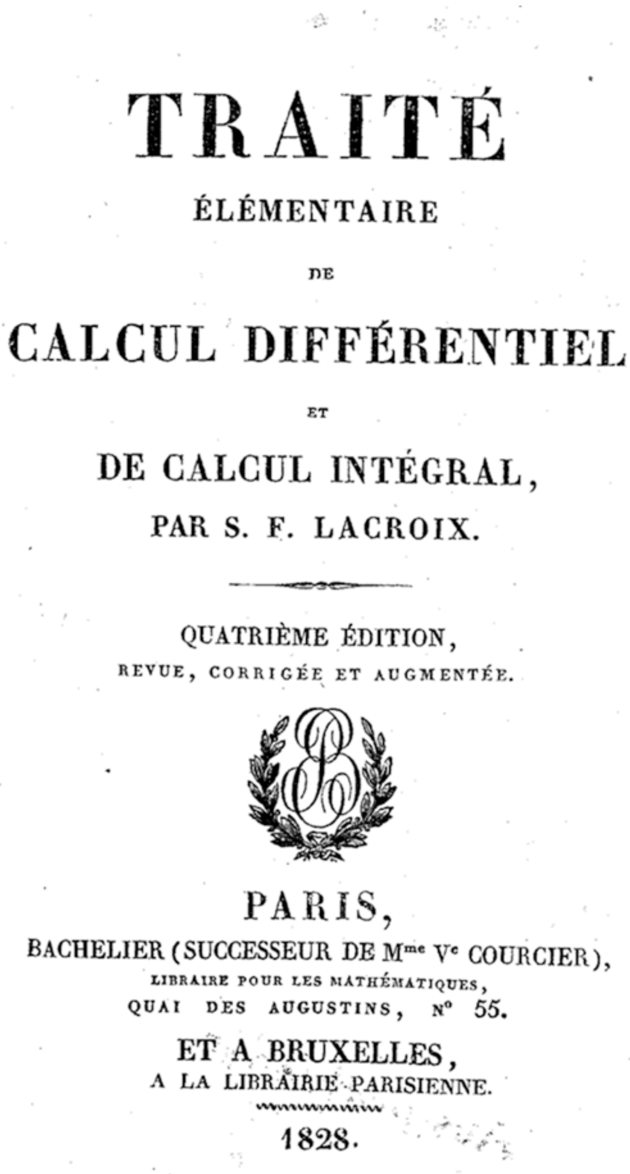
His career seems to have started after the *Thermidor* and the fall of Robespierre. His first contribution to reorganising the school system was his appointment as a member of the jury established to decide on the concours for *livres élémentaires* in 1794. Lacroix was obviously committed to the *Idéologues* faction. In 1795, he was Monge's assistant in his lectures on descriptive geometry at the *École Normale*. Soon after, he obtained a post in the government department for public instruction. During the same period, he became a mathematics teacher at one of the *Écoles Centrales* in Paris, then named *École Centrale des Quatre Nations*; later he became professor of analysis at the *École Polytéchnique*, and later *Examinateur Permanent* at this institution which dominated higher education. While also a member of the *Institut*, the name of the Academy of Sciences then, he became dean of the *Faculté des Sciences* de Paris, established in 1809 as part of Napoleon's *Université Impériale*. Eventually, he even obtained the position of professor at the renowned *Collège de France*.

Lacroix can be seen as a prototype and a first performer of the programme of *livres élémentaires* designed to restructure the mathematical knowledge taught according to the most advanced scientific inventions. He managed to perform such a programme impressively for Differential and Integral Calculus.

Obviously, he did not belong to the "inventors", the leading scientists, who since d'Alembert had traditionally been assigned the task of writing *livres élémentaires*. In 1796, after the failure of the concours, the *Idéologues* as a group seem to have been willing to re-evaluate this usual procedure, as is clear from a critique of the reprint of J. A. J. Cousin's calculus textbook that was published in the *Décade*, the most important periodical of the group of *Idéologues*.

Although the anonymous author of the review made a strict distinction between users of popular *livres élémentaires*, intended for general education, and intellectuals who wanted to "deepen the science" and who would do better by reading books written by the "inventors", the reviewer was willing to confer the rank of an inventor to a textbook author, provided he was able to present his elements in the best order and in the simplest and clearest manner, and to detach the science from its technical aspects, maintaining a graduated exposition so as to always inform students of where they have reached actually (*Décade*, n. 76, 30 Prairial, ano IV, 18 de junho de 1796, p. 517).

One year later, in 1797, in its report on Lacroix's project for a new textbook on Differential and Integral Calculus (Fig. 7.1), the French Academy (*Institut de France*) highlighted the intrinsic relationship between progress in research and clarity in its foundations (albeit no longer assigning a privileged role to inventors):


TRAITÉ
ÉLÉMENTAIRE
DE
CALCUL DIFFÉRENTIEL
ET
DE CALCUL INTÉGRAL,
PAR S. F. LACROIX.

QUATRIÈME ÉDITION,
REVUE, CORRIGÉE ET AUGMENTÉE.

PARIS,
BACHELIER (SUCCESSEUR DE M^{me} V^{e} COURCIER),
LIBRAIRE POUR LES MATHÉMATIQUES,
QUAI DES AUGUSTINS, N° 55.
ET A BRUXELLES,
A LA LIBRAIRIE PARISIENNE.
1828.

Fig. 7.1 Cover of Lacroix's textbook version of his Analysis Compendium, fourth edition of 1828

To present difficult theories with clarity, to connect them with other known theories, to dismantle some of the systematic or erroneous parts which might have obscured them at the time of their emergence, to spread an equal degree of enlightenment and precision over the whole; or, put shortly, to produce a book which is at the same time elementary and up to the mark in science, this is the objective which Citizen Lacroix has taken to himself and which

> he could not have attained without engaging himself in profound researches and by progressing often at the same level as the inventors. (quoted from Schubring 1987, p. 43.)

The failure of the original programme of *livres élémentaires* thus led to a new perspective on textbook authors. They were then seen as specialists in their own right, different from leading scientists, and with even a high status.

7.3 The "Common" Knowledge

Lacroix also presents a pertinent example to analyse the originality of textbook authors. As is well known, many authors of these books tended to copy material from earlier textbooks on a large scale and tried to keep this fact a secret.

One exception to this rule is Lacroix, who recognised at various instances the loans he made. In the first edition (1797) of his *Traité élémentaire d'arithmétique*, he frankly stated that the book was "in considerable degree the work of Citizen Biot, a teacher of mathematics at the *École Centrale of the département de l'Oise*". (Lacroix 1797, vol. I, p. xj). In the second edition, from 1800, these words were repeated (but no longer continued in later editions). Similarly, Lacroix credited a predecessor in the first edition of his 1799 algebra textbook:

> Urged by the shortness of time which does not allow me to completely compose a treatise of algebra in the time that remains before I need it. I have completed the notes and additions which I had inserted into the fifth edition [of Clairaut's algebra, G. S] by new articles or by pieces selected from Bézout's algebra, and I have done this in such a manner that a coherent whole has emerged All that has been taken from Bézout's algebra has been put between brackets (Lacroix 1799, pp. 1–2)

In fact, the parts copied from Bézout were indicated with brackets. Calculating the total of these parts (borrowed only "to make the book complete"), I got more than three-quarters of the entire book! The second edition, from 1800, tried to disguise the "common ownership" model with clever wording:

> [In the first edition] I borrowed from the third part of Bézout's textbook series some articles which deal only with details of operations and which are common to all books and all methods. I did so in order to fill the gaps which were left between the notes and the complements, by which I had accompanied my edition of Clairaut and in which the most subtle aspects of algebra were discussed (Lacroix 1800, p. xv)

Lacroix specified that his revision of these virtually trivial parts was less thorough than that of his own "subtleties", adding that he would henceforth omit the revealing brackets. We can thus infer that two of Lacroix's textbooks: the *Traité élémentaire d'arithmétique* and the *Éléments d'Algèbre* were not really his own production but a work copied on a large scale from other authors (Jean-Baptiste Biot and Étienne Bézout). Despite this gentlemanly attitude towards "common ownership", Lacroix used aggressive strategies to control the market, which is most visible in his rivalry with his competitor Legendre in geometry.

7.4 The Quarrel with Legendre About the Market

In February 1799, Legendre learned that Lacroix was going to publish a schoolbook on geometry. He was seriously concerned that his own geometry book, hitherto the only modern French textbook, might be threatened by a dangerous competitor. Thus, Legendre proposed to Lacroix a meeting in which he pressured him to desist from publishing the planned geometry volume. Lacroix agreed and promised he would desist; however, this would not be his last word on the matter. Three days later, Duprat, Lacroix's editor (and also of many other important mathematics books) went to meet Legendre: Lacroix had not only given up the geometry text, he had also withdrawn his arithmetic and algebra textbooks. Duprat complained to Legendre about the enormous economic loss he would incur from withdrawing three works at once. Legendre responded with a letter to Lacroix, dated February 16, 1799, in which he reaffirmed his commitment to smooth Lacroix's abandonment of his own project: Lacroix should continue to use Legendre's geometry textbook in his position as a teacher at an *École Centrale*, but he was free to "complement" this text "orally" and "to develop the subject via other works" in his own writing. As potential examples for that, Legendre listed trigonometry, arithmetic, and algebra, "which I never intended to write about". But as "the sacrifice you made on my behalf" (keeping Legendre's geometry as the schoolbook at the *École Centrale*) created too many problems for him due to Duprat's pressures, Legendre now proposed an oligopoly in place of his own previous monopoly:

> But since this sacrifice costs you too much, and since it is too expensive for Citizen Duprat, and since it is better that among three concerned persons only one is sacrificed instead of two, I gladly agree that I am the one to be sacrificed. I therefore regard the promise which you gave me three days ago as not having occurred and give you liberty to publish your own geometry (quoted from Schubring 1987, p. 46)

Lagrange's activities in the 1799 deliberations of the *Conseil d'instruction publique* provide the key to understanding why Lacroix insisted on publishing his own geometry textbook as well, and why he got Duprat to intervene on his behalf with Legendre: Lagrange's criterion for choosing a textbook was whether it was "complete", that is, whether it covered all the traditional subdisciplines of mathematics. In late 1798 and early 1799, Lacroix appears to have been extraordinarily pressed to publish his own schoolbook series. His first book, from 1795, had been on descriptive geometry, and the next, from 1797, on arithmetic and algebra, but the one on algebra had been a mere reprint (of Clairaut's), which made him decide to publish another (apparently) "proper" one, in 1799. The remaining gap in his programme to achieve a *Cours*, however, concerned geometry – the more traditional school subject, and – in view of Lagrange's criterion, which was certainly known to Lacroix even before concrete deliberations in the Conseil had begun – the latter hastened to fill the fatal gap. It is easy to imagine Lacroix's anxiety, given the prospect that Legendre might create a situation in which he himself would be left with an incomplete *Cours*! And most likely Lacroix was a behind-the-scenes conspirator in the

Conseil's manoeuvres against Lagrange's no-choice policy, willingly offering with glee the information that his own *Cours* would also be "complete" soon.

With the exception of his compendium on differential and integral calculus, which was a highly original work, Lacroix's authorship is a delicate matter in other cases. His descriptive geometry was published after he had been Monge's assistant in his first public lectures on descriptive geometry at the *École Normale*.[4] And as for his arithmetic and algebra, I have already mentioned that they had been written by Biot and Bézout. In this way, questions can also be asked about the originality of his geometry book. Lacroix himself provided evidence on this matter in an extensive letter to Legendre, after the latter had given him the "freedom" to publish his planned geometry book. In that letter, Lacroix tried to justify why he had also proposed to write a book on geometry, giving its general lines while affirming that his plans and studies in relation to it were already old and not motivated by the current situation. In addition, he tried to defend himself against accusations of being overly prolific or trying to get rich.

In that letter, whose draft still exists, Lacroix admitted that he had always used, in his previous positions as a teacher, Bézout's geometry book, adding that it had not been "difficult (for me) to observe its deficiencies". On the other hand, he showered Legendre with praise for his book, which he claimed to have bought immediately, in particular for its rigour:

> yours [geometry] seemed to me the most carefully written and the most severe in method.[5]

Lacroix claimed to have adopted it as a teacher at the *École Centrale*. However, after he had started teaching, he had begun to think more about teaching methods, and had taken note of observations and experiences from his own classroom. From these notes, the structure of his own textbook had emerged, which would express his methodological points of view. The project had been motivated initially by a certain idiosyncrasy of taste in the fundamental aspects, since anyone in general was more convinced about his own ideas, but after rethinking, he had begun to criticise the method of Legendre's book. Lacroix explained this: the architecture of Bézout's geometry had been for him more natural than that of Legendre, for example, in the use of theorems about surfaces to prove propositions about proportional lines. And he added that he had preferred to put problems after the theorems on which they depended, claiming to have adopted a more natural order (en m'attachant ainsi à un ordre que je regarde comme plus naturel)[6]; he still claimed to have preserved the best of several classical authors, not only of Bézout and Legendre, but even of

[4] Barbin refutes plagiarism attributed to Lacroix for this textbook, relating that according to Jean-Nicolas-Pierre Hachette (1769–1834) Lacroix had written most parts of the book before 1795 (Barbin 2019, p. 12).

[5] la vôtre [géométrie] m'a paru la plus soigneusement écrite et la plus sevère dans la méthode. Draft of a letter by Lacroix to Legendre, no date (about February 1799), Bibliothèque de l'Institut, Paris: Papiers de S.-F. Lacroix, mss. 2397.

[6] Following thus an order which I regard as more natural.

Euclid and the Greeks in general (les auteurs anciens).[7] Thus, he had given a more "analytical" presentation of geometry (ibid.).

After having received a copy of Lacroix's geometry, Biot, in a letter to Lacroix, actually praised it as a perfected realisation of the analytic method, saying that it "prepared the intellect for the search for truth" better than Legendre's book (see Schubring 1987, p. 46).

7.5 The Cultural and Epistemological Pressures: The Case of Algebra

Schoolbook authors need to take into account social and cultural changes and their impact upon teaching content and methods. Lacroix, though influential, had to make concessions to such pressures. To explain that, we are led here to discuss more concretely his algebra book, already mentioned several times. In fact, Lacroix published several versions – and these corresponded to general changes in pedagogical and epistemological conceptions within French culture.

The algebra book, which in Lacroix's lifetime had 17 editions in France, underwent decisive changes in structure and content in its first four editions, from 1797 to 1803. The revisions are characterised by a growing distance from Clairaut's point of view, having propagated the inventor's method, and by a growing distance also from the Revolution's point of view regarding algebraisation, finally adhering to the epistemological setback that affirmed the primacy of geometry.

Of the first four editions, we will call the first one: as "edition number zero" because Lacroix did not claim authorship and preferred to publish it anonymously. Only the abbreviation "L. Ç." at the end of the "Avertissement de l'éditeur" (Editor's Notice) suggests the identity of the editor. This "zero edition" is titled:

0. edition: Éléments d'Algèbre, par Clairaut, cinquième edition. Avec des Notes et des Additions tirées en partie des Leçons donnés à l'École Normale par Lagrange et Laplace, et précédée d'un Traité Élémentaire d'Arithmétique. Paris, chez Duprat, an V = 1797.

Why did Lacroix not choose to write his own text and why did he decide to re-edit Clairaut's work? Here we can observe the impact of the diffusion of d'Alembert's ideas regarding the inventors' method as the best method for schoolbooks. As there already existed, in the case of Algebra, a book that claimed to realise this methodical principle, Lacroix decided himself to adopt it as the basis for his own book. Lacroix emphasised this methodological principle of adhering to the inventor's method:

> Clairaut's Elements of Algebra, in which readers take part, in a way, in the invention of Analysis, contains many and varied applications; they also suppose very little knowledge of

[7] The ancient authors.

> arithmetic, and this advantage is precious to many readers who have been discouraged[8] by the study of this part in somewhat extended treatises. These are the reasons which, together with the celebrity of the Author, have given such a great reputation to this work (Lacroix 1797, p. vij).[9]

Lacroix already expressed some reservations in this edition: despite all its excellence, he remarked that Clairaut's book contained only a small part of the useful knowledge of algebra, and thus needed to be supplemented:

> This work [...] very far from comprising everything that can be useful in algebra: they [these reasons] have led us to believe that it could be regarded as an excellent preliminary that only needed to be completed (Lacroix 1797, p. vij).[10]

Lacroix did not attribute to himself any of the additions he had deemed necessary, crediting them to famous mathematicians such as Euler, Lagrange and Laplace:

> The additions that should be made are found, for the most part, in the writings of Euler, in those of Lagrange, and in the lessons he gave at the *École Normale*, jointly with Laplace (Lacroix 1797, pp. vij–viij).[11]

Actually, Lacroix distinguished between two types of additions: first, simple notes that summarised particular results into more general formulations – thus already implying a critique of a problem-oriented one-sided approach that did not generalize its findings; and second, theoretical developments that were absent from Clairaut's version:

> These additions are of two kinds; some are simple notes whose purpose is to summarise the rules scattered in the text and to give more general demonstrations of some of them than those of the author; the others contain the developments of some theories of which Clairaut did not speak (Lacroix 1797, p. viij).[12]

[8] It is revealing that Lacroix still refers here, at the beginning of the first national and public education system, to the old methodological guideline "ne pas rébuter" (not to discourage), which is characteristic for the degenerated textbook triangle, the student dominating the textbook, without regard to a teacher.

[9] Les Éléments d'Algèbre de Clairaut, dans lesquels les lecteurs prennent part, en quelque sorte, à l'invention de l'Analyse, renferment des applications nombreuses et variées; ils supposent d'ailleurs fort peu de connoissances en Arithmétique, et cet avantage est précieux à beaucoup de lecteurs que l'étude de cette parti dans des traités un peu étendus, a rebutés. Ce sont ces raisons qui, jointes avec la celebrité de l'Auteur, ont donné une si grande réputation à cet ouvrage.

[10] Cet ouvrage [...] fort éloigné de comprendre tout ce qui peut être utile en Algèbre: elles [ces raisons] ont fait croire qu'on pouvoit le regarder comme un excellent préliminaire qu'il ne s'agissoit de completer.

[11] Les additions qu'il convenoit d'y faire se trouvent, pour la plus grande partie, dans les écrits d'Euler, dans ceux de Lagrange, et dans les leçons qu'il a données à l'École Normale, conjointement avec Laplace.

[12] Ces additions sont de deux sortes; les unes sont de simples notes qui ont pour objet de résumer les régles éparses dans le texte, et de donner de quelques unes des démonstrations plus générales que celles de l'Auteur; les autres renferment les développements de quelques théories dont Clairaut n'a point parlé.

The second edition of Lacroix's algebra book – published only 2 years later, denoted by him as its first edition, constitutes in fact the first, since Lacroix attributed now parts of the text to himself, and even some of those parts that he had attributed in the "zero edition" to other famous scientists:

1. edition: [S. F. Lacroix] *Élémens d'Algèbre.* Paris, an VIII (1799), chez Duprat.

This edition shows a clear break with d'Alembert's notion of the inventors' method: Lacroix eliminated Clairaut's text, replacing it with another, but not yet one of his own authorship. He chose Bézout's algebra as his base, supplementing it with notes and additions he had inserted, such as texts by Euler, Lagrange, and Laplace – in the "zero" edition, and with "some new articles" written by himself. Lacroix justified his elimination of Clairaut's texts because they were too long, and he praised, on the contrary, Bézout's "rapidity". Lacroix admitted the lack of rigour in Bézout's demonstrations and claimed to have corrected these flaws. Lacroix also apologised for not having written an entire schoolbook by himself, saying he didn't have time for it, needing to use it in his own work as a teacher. The choice of Bézout was certainly motivated by the popularity of his book at the time (see Chap. 6).

> Pressed by time, which does not allow me to write an entire *Traite d'Algèbre* by the time I need it and not wanting to reprint the text of *Élémens d'Algèbre* by Clairaut, which contains many lengths, I have completed, either by new articles, or by extracts from Bézout's *Algèbre*, and in such a way as to form a single whole, the notes and additions that I had inserted in the 5th edition of the first of these works, and which the public favorably received. The rapidity of Bézout's exposition and the purity of his writing have generally caused his course to be esteemed; but also all those who value exactitude in the demonstrations, desired the changes which they will find here, and will perhaps see with pleasure algebra presented according to the method of invention, freed from the meticulous details which overload it (Lacroix 1799, pp. 1–2).[13]

At that time, the notion of the inventors' path still seems to have been seen as a positive reference, as Lacroix mentioned, claiming to have modified its original extent. As he had abandoned Clairaut's text and as Bézout neither claimed nor actually represented such an approach, the last statement in the quote was without substance. In the third edition, denoted as his second by Lacroix, published just 1 year later – but still without an explicit authorship, Lacroix now openly criticised the

[13] Pressé par le temps qui ne me permet pas d'écrire en entier un Traité d'Algèbre d'ici à l'époque où j'en aurait besoin, et ne voulant pas réimprimer le texte des Élémens d'Algèbre de Clairaut, qui contient beaucoup de longueurs, j'ai complété, soit par des articles nouveaux, soit par des morceaux extraits de l'Algèbre de Bézout, et de manière de former un seul tout, les notes et les additions que j'avois insérées dans la 5e édition du premier de ces ouvrages, et que le public a favorablement accueili. La rapidité de la marche de Bézout et la pureté de sa rédaction, ont fait généralement estimer sons cours; mais aussi tous ceux qui font cas de l'exactitude dans les démonstrations, y désiroient des changemens qu'ils trouveront ici, et verront peut-être avec plaisir l'Algèbre presentée suivant la méthode d'invention, degagée des détails minutieux qui la surchargent.

methodology of following the inventors' path dominant until then, granting it only a limited value, as a motivation for introducing a schoolbook:

2. edition: [S. F. Lacroix], *Élémens d'Algèbre. Seconde édition, revue et corrigé*. Paris, chez Duprat, an IX (1800).

Although the text was essentially identical to the previous edition, Lacroix devoted most of its new preface to explaining the shortcomings of the inventors' path, in general, and of Clairaut's textbook, in particular:

> Clairaut does not keep the march of invention within the limits it deserves: certainly, this march is necessary to enlighten and to encourage those who begin the study of algebra, but it becomes meticulous, and is overloaded with details when one pursues it rigorously beyond the first notions (Lacroix 1800, p. vij).[14]

Although using his reprint of Clairaut, Lacroix told he had convinced himself of the tedious prolixity of this method and that its value is restricted to the first basic notions:

> I did not hesitate in convincing myself that it was necessary to restrict the march of invention a lot, and that, when the student had overcome the first difficulties, that he perceived the goal of science, that applications have convinced him of the usefulness of his work, all what is needed to encourage him to continue is to present the materials to him in the order in which they arise from one another.
>
> I therefore believed that I had to restrict myself to the march of invention only to introduce the elements (Lacroix 1800, p. ix).[15]

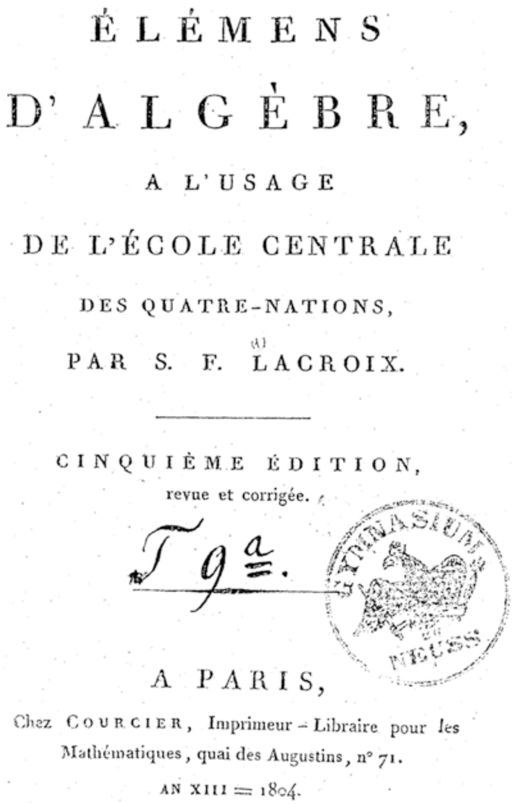
The fourth edition, officially Lacroix's third edition, followed just 3 years later, in 1803, and, for the first time, contained a text apparently written exclusively by Lacroix himself – and naming him as the author (Fig. 7.2):

3. edition: S. F. Lacroix, *Élémens d'Algèbre, à l'usage de l'École Centrale des Quatre-Nations. Troisième édition, revue et corrigée*. Paris, chez Courcier, an XI, 1803.

[14] Clairaut ne tient pas la marche d'invention dans des justes bornes: certes, cette marche est nécessaire pour éclairer et pour encourager ceux qui commencent l'étude de l'Algèbre, mais elle devient minutieuse, et se trouve surchargé de détails lorsqu'on la poursuit rigoureusement au-delà des premières notions.

[15] Je ne tardait pas à me convaincre par moi-même qu'il était nécessaire de resserrer beaucoup la marche d'invention, et que lorsque l'élève a passé les premières difficultés, qu'il a apperçu le but de la science, que des applications l'ont convaincu de l'utilité de son travail, il ne faut plus, pour l'engager à continuer, que lui présenter les matières dans l'ordre où elles naissent les unes des autres.

Je crus donc devoir ne m'astreindre à la marche d'invention que pour faire l'introduction des Élémens.



ÉLÉMENS

D'ALGÈBRE,

A L'USAGE

DE L'ÉCOLE CENTRALE

DES QUATRE-NATIONS,

PAR S. F. LACROIX.

CINQUIÈME ÉDITION,

revue et corrigée.

A PARIS,

Chez COURCIER, Imprimeur-Libraire pour les Mathématiques, quai des Augustins, n° 71.

AN XIII = 1804.

Fig. 7.2 Cover of Lacroix's algebra, fifth official edition of 1804

From this new edition on, Lacroix had not only bypassed Clairaut, but now also Bézout's text, replacing it with a text he composed himself.[16] No trace remained of the formerly so revered method of the inventors. After several years' experience running a public-school system, Lacroix had realised that such a "genetic" method was not applicable to public instruction; in order to better generalise knowledge, Lacroix now preferred the direct development of theories and their applications, an approach he called "dogmatic": "la forme dogmatique, nécessaire pour rendre aisée la pratique des règles" (the dogmatic form, necessary to make the practice of rules easy) (Lacroix 1803, p. vj).

Here, Lacroix maintained that he had "borrowed" from Bézout in the earlier editions only several "shreds":

[16] In previous publications, I attributed this rupture to the seventh edition, of 1807, but it had already occurred in the third, of 1803. Its text has been essentially preserved in the numerous subsequent re-editions – no new conceptual ruptures did occur.

> I completely replaced the various pieces that the short time I had to give to his first writing had forced me to borrow from Bézout's (Lacroix 1803, p. v).[17]

The main reason given by Lacroix for dismissing Bézout and making a presentation of his own was the fact that he found it necessary to reorganise the entire exposition due to a profoundly revised conception of negative quantities. The series of Lacroix's algebra schoolbooks represents, therefore, not only a revealing case of modifications in didactic-pedagogical conceptions according to changes in social and cultural viewpoints, but it also shows the influence of changing epistemological conceptions. In fact, 2 years earlier, in 1801, Lazare Carnot had published the first version of his own conception of rejecting negative quantities and reinterpreting them in geometric notions. This first publication on the correlation of figures, from 1801, was supplemented by Carnot in his *Géométrie de Position* in 1803. Carnot's return to a synthetic method, privileging geometry as the sole source of mathematical truth, soon became popular, and Lacroix, agreeing with Carnot, immediately applied many of his ideas to a complete restructuring of his own algebra schoolbook.

In France, throughout the eighteenth century, the prevailing conception of negative numbers had been to accept these quantities – somewhat ambiguously – as real quantities; they were admitted without restrictions in algebraic operations, but certain limits were imposed by different authors in relation to the acceptability of negative solutions (see Schubring 2005, pp. 102 ff.). Lazare Carnot, however, had proposed a radical break with this consensus in two publications, in 1801 and 1803 (see ibid., pp. 355 ff.). Carnot refused a separate status to algebra, and confined its meaning largely to that of arithmetical operations; he gave geometry the most fundamental domination and status in mathematics. Algebraic operations were only admitted when interpretable in geometric terms. The only legitimate objects in arithmetic and algebra were absolute numbers, that is, numbers that were legitimated by some substantialist meaning. Consequently, Carnot restricted algebraic operations to "executable" cases; for example, the distributive law.

$$(a-b)c = ac - bc$$

had to be restricted to the case where $a > b$.

And the thus restricted domain of algebra was converted by Carnot into a geometric calculus, based on oriented lines:

> From which I conclude, 1^0. that every isolated negative quantity is a being of the mind, and those met in calculations are just simple algebraic forms incapable of representing any real and actual quantity (Carnot 1803, p. xviij; transl. quoted from Schubring 2005, p. 359).[18]

[17] J'ai remplacé entièrement les divers morceaux que le peu de temps que j'avais eu à donner à sa première rédaction m'avait forcé d'emprunter de celui de Bézout.

[18] De-là je conclus, ... que toute quantité négative isolée est un être de raison, et que celles qu'on rencontre dans le calcul, ne sont que des simples formes algébriques, incapables de réprésenter aucune quantité réelle et effective.

> The geometry of position is, therefore, properly speaking, the doctrine of quantities called positive or negative, or rather the means of providing them; for that doctrine is entirely rejected here (ibid, p. ij; transl. quoted from ibid., p. 357).[19]
>
> I would say that the geometry of position is that where the notion of isolated positive and negative quantities is supplemented by that of direct and inverse quantities (Carnot 1803, p. xxxiv).[20]

This rejection of the generalising tools of algebra and the return to substantialist interpretations of geometric notions became generally accepted in France within a few years. This provoked a rupture, particularly with regard to school mathematics. In fact, Lacroix's algebra book which became mandatory for all secondary schools in France from 1803 onwards was exactly that profoundly revised edition.

Until the second edition, in 1800, Lacroix had continued to adhere to Bézout's notions with his ambiguous acceptance of negative numbers as legitimate mathematical objects, but reinterpreting equations when they produced a negative solution (Schubring 2005, p. 125 f.) – this reinterpretation had been first proposed by Clairaut in 1746. The third 1803 edition (maintained in its numerous subsequent reprints), however, marked a radical change: the existence of negative quantities was no longer admitted, there was no reflection on the related operations; rather, negative solutions, when mentioned, were qualified as *absurdité* (an "absurdity"). Lacroix justified his rewriting as a result of didactic reasons, arguing that notions as complicated as negative numbers should not be placed at the beginning of an algebra schoolbook – a position that he judged had forced most authors to introduce the concept not in an explanatory way but by exposing it via rules, thus requiring to learn them by memory:

> Considering isolated negative quantities was generally placed too close to the beginning in most *livres élémentaires*; [...]. Also, most of the authors have exposed this subject by relying on memory (Lacroix 1803, p. vj).[21]

An analysis of this revision reveals two epistemological reasons for such a turnaround:

– the subtraction operation is restricted to the case of a positive "remainder" (Lacroix 1803, p. 92);
– the underlying notion of quantities proves to be that of concrete quantities, and not of abstract quantities or numbers; in particular, amounts considered in equations are *francs,* therefore, negative *francs* did not make sense, effectively. For example, a more than two-page long discussion on linear system of two equations with two apparently abstract quantities, x and y, such as $60x + 7y = 46$, ends

[19] La géométrie de position est donc, à proprement parler, la doctrine des quantités positives et négatives, ou plutôt le moyen d'y suppléer, car cette doctrine y est entièrement rejetée.

[20] Je dirais que la géométrie de position est celle oú la notion des quantités positives et négatives isolées, est supplée par celle des quantités directes et inverses.

[21] La considération des quantités négatives isolées était en général placée trop près du commencement dans la plupart des livres élémentaires; [...]. Aussi, la plupart des auteurs se sont adressés sur ce sujet à la mémoire.

abruptly by revealing the solution given in quantities, namely in *francs,* and not in abstract numbers: "$x = 5$ fr, $y = 2$ fr" (ibid., p. 88). Lacroix proudly claimed to have exposed the true metaphysics of mathematics by means of this revision, thus confirming Carnot's influence:

> I flatter myself [...] that one will thus grasp here the metaphysics of calculating, which seems to me to have been given nowhere in a satisfactory way, and without which algebra seems only a genuine handicraft, devoid of all interest for thinking heads (Lacroix 1803, p. viij).[22]

It is highly revealing that this epistemological break failed to be accepted in Germany; and a German translation of this seventh edition of Lacroix's algebra, published in 1811 by Mathias Metternich (1747–1825), who had taught Mathematics at the *Lycée* in Mainz according to Lacroix's books, was truly converted into a complete rebuttal of the original, by his annotations, comments, and even direct omissions and changes of the text (Schubring 1996). Metternich, originally a mathematics professor at the tiny university of Mainz, was active in research on the theory of parallels, and familiar with the French mathematics scene, since he had stayed in Paris and Alsace in 1794 and 1795.

While one had paid no attention in Germany to the earlier French debates about the nature of negative quantities, Carnot's work was immediately discussed – and refuted (Busse 1804). The heated discussions were unanimous in rejecting his epistemology and in defending the legitimacy of negative numbers. Right in the preface of his translation, Metternich emphasised that Lacroix's notions of the signs plus and minus were fluctuating and that his presentation of the different cases of the use of the signs plus and minus lacked mathematical precision. After introducing the subtraction operation, Metternich explained in footnotes that Lacroix's proofs were not rigorous, showing how they had to be transformed in order to arrive at generally valid proofs. Soon, Metternich reached a point where footnotes no longer were sufficient; he began to insert entire paragraphs and even brief chapters in order to introduce a general notion of negative numbers. Thus, he declared that continuing the discussion of more parts of Lacroix's text to be "fussily long" and, eventually, ceased translating:

> I have ceased translating this long chapter [...] since the reader [after reading my insertions] will no longer doubt the theory of subtraction and of multiplication (Metternich 1811, p. 121.).[23]

[22] Je me flatte[...] qu'on y saisira encore la métaphysique du calcul, qui me semble n'avoir eté donnée nulle part d'une manière satisfaisante, et sans laquelle l'Algèbre ne paraît qu'un véritable métier, dénué de tout intérêt pour les têtes pensantes.

[23] Ich habe diesen langen § des Autors nicht weiter übersetzt [...] indem ich glaube, daß [Verweis auf die gemachten Einfügungen] kein Zweife1 mehr über die Theorie der Subtraktion und über die der Multiplikation statt findet.

7.6 Methodological Comments

Lacroix himself had emphasised the interrelationship between basic mathematical theories and other branches of mathematics, not only for applying their fundamental concepts but also for developing their meaning; in particular, he had pointed out that a perfect understanding of the concept of negative quantities was provided by its development within geometry, using coordinates:

> It is moreover only by the application of algebra to geometry, that one can conceive the theory of negative quantities in all its extent (Lacroix 1803, p. vj).[24]

His suggestion underlines the necessity of contextualised analyses as discussed in the introduction of the book, namely, that an analysis of textbooks cannot be restricted to the respective mathematical theory or branch in isolation; on the contrary, analysis has to consider the interconnections of mathematical theories and study the application of concepts developed in one branch to other branches of mathematics. This reinforces the fruitfulness of the three-dimensional approach proposed to the textbook analysis (see Chap. 1).

7.7 Translations of Lacroix's *Cours*

The textbooks Lacroix have published were enormously successful and had a notable influence not only in France but also in many other countries – in Europe and in North and South America. In fact, his works have been translated into many languages, and in some cases even several times: for example, the elementary treatise on differential and integral calculus has been translated twice into German, and the algebra book three times. One of these translations, actually, was the refutation by Mathias Metternich. An 1822 Dutch translation was also a rebuttal (cf. Beckers 2000).

In Brazil, Lacroix's influence was particularly strong; already in the first 3 years after the eventual introduction of a printing press, five translations were published.

> TRATADO ELEMENTAR D'ARITHMETICA POR LACROIX, TRADUZIDO DO FRANCEZ POR ORDEM DE SUA ALTEZA REAL O PRINCIPE REGENTE NOSSO SENHOR. Para uso da Real Academia Militar, e accrescentado com taboas para a reducção das medidas Francezas antigas e modernas entre si, a medidas Portuguezas, e reciprocamente, POR FRANCISCO CORDEIRO DA SILVA TORRES, Sargento Mór do Real Corpo d'Engenheiros, e nomeado Lente da mesma Academia. RIO DE JANEIRO, 1810. NA IMPRESSÃO REGIA. Por Ordem de S.A.R.
>
> ELEMENTOS D'ALGERA POR Mr. LA CROIX TRADUZIDOS EM PORTUGUES, POR ORDEM DE SUA ALTEZA REAL O PRINCIPE REGENTE NOSSO SENHOR, PARA USO DOS ALUMNOS DA REAL ACADEMIA MILITAR DESTA CORTE, POR Francisco Cordeiro da Silva Torres, Sargento Mór do Real Corpo de Engenheiros, e Lente da mesma Academia. RIO DE JANEIRO. 1811. Na Impressão Regia. Por Ordem de S.A.R.

[24] Ce n'est d'ailleurs que par l'application de l'Algèbre à la Géométrie, qu'on peut concevoir dans toute son étendue la théorie des quantités negatives.

TRATADO ELEMENTAR DE APPLICAÇÃO DE ALGEBRA Á GEOMETRIA POR LACROIX, TRADUZIDO DO FRANCEZ, ACCRESCENTADO, E OFFERECIDO AO ILLUSTRISSIMO E EXCELLENTISSIMO SENHOR D. JOÃO D'Almeida de Mello de Castro, conde das galveas Conselheiro, Ministro e Secretario d'Estado dos Negocios da Marinha e Dominios Ultramarinos encarregado interinamente da Repartição dos Negocios Estrangeiros e da Guerra Gran-Cruz das Ordens de S. Bento d'Aviz, e da Torre e Espada, Commendador da de Christo &c. &c. &c. POR JOSÉ VICTORINO DOS SANTOS E SOUZA, Capitão graduado do Real Corpo d'Engenheiros, Lente da Real Academia Militar. RIO DE JANEIRO. NA IMPRESSÃO REGIA. 1812. Por Ordem de S.A.R.

ELEMENTOS DE GEOMETRIA POR LACROIX. POR JOSÉ VICTORINO DOS SANTOS SOUZA. 1812.

TRATADO ELEMENTAR DE CALCULO DIFFERENCIAL, E CALCULO INTEGRAL POR Mr. LACROIXÇ POR ORDEM DE SUA ALTEZA REAL, TRADUZIDO EM PORTUGUEZ PARA USO DOS ALUMNOS DA REAL ACADEMIA MILITAR DESTA CORTE, POR FRANCISCO CORDEIRO DA SILVA TORRES, Sargento Mór do Real Corpo de Engenheiros, e Lente da mesma Academia. PARTE 1A. CALCULO DIFFERENCIAL. RIO DE JANEIRO. NA IMPRESSÃO REGIA, 1812. Por Ordem de S.A.

Chapter 8
Textbook Versus the Autonomy of the Teacher: The Prussian Case

8.1 Independence of the Teachers

As outlined in Chap. 5, Prussia represents an entirely different case in terms of realising the new alignment between teaching and research. In contrast to France, where the centralised educational system led to a preference for textbooks and literalism, Prussia – and to some extent other German states as well – adopted a policy that favoured the role of the teacher and consequently that of orality, along with the acceptance of a creative function of the "research imperative" in Wilhelm von Humboldt's sense: teaching can induce a reconsideration of foundations, thus promoting the progress of science. Therefore, we have an emblematic case for the textbook triangle where the teacher dominated the textbook, due to this new function of orality.

In Prussia, educators were well aware of this contrast, particularly of its political dimensions. When large territories of the Rhineland that had been under French rule since the late 1790s fell into Prussian power after Napoleon's defeat in 1814, the new governor of the province immediately abolished the textbook recommendation by the authorities:

> Uniformity of textbooks in schools is good for nothing; it curtails the teachers' freedom of method and hinders or impedes progress to improvement; therefore, the one-sided decree of the French *Université*, which prescribed the same textbooks everywhere, is herewith abolished (Schubring 1988a, p. 10).[1]

[1] Eine Gleichförmigkeit der Lehrbücher in allen Schulen taugt nichts; sie schadet der Freiheit in der Methode des Lehrers, und beschränkt oder verhindert das Fortschreiten zum Bessern; daher wird die einseitige Verordnung der französischen Universität, welche überall einerlei Lehrbücher vorschreibt, hiermit aufgehoben (quoted from Schubring 1991a, p. 182).

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_8

The schoolbooks should henceforth be chosen by the director of each Gymnasium, after deliberation with the collegiate. Such an emphasis on teacher independence was an expression of the then prevailing spirit of Neohumanism,[2] which tended to encourage autonomy of the citizen. In 1814, the Prussian Ministry of Instruction issued a decree that emphasised the liberties of teachers, generalising the decree of the Rhineland's governor:

> It does not correspond to the principles of a rational administration of the school system to prescribe generally the use of certain textbooks in schools. This would render more difficult the introduction of better textbooks published by and large and thus also the improvement of school procedure. [...] We thus refrain from every kind of positive prescription of certain textbooks for schools. [...] Our restraint in taking positive action, however, will not prevent negative governmental action aimed at removing the unsuitable (ibid.).[3]

Actually, such "negative prevention" was only very rarely applied.

In the spirit of Neohumanism, secondary school teachers ("Gymnasien"), in view of their scientific studies in the modernised Prussian universities, were also considered as scholars, consequently being treated with the same type of autonomy in their teaching and with the same freedom as to methodology that university professors had been using for some time.

One explanation for this non-centralising educational system is given by Germany's peculiar political structure: before 1806, and until the end of the Holy Roman Empire, Germany had consisted of hundreds of individual territories. Consequently, the nation had not just one cultural centre, but an impressive variety of regionally based cultural centres which enhanced competition for self-styled cultural and scientific production. This spirit of regionality was also typical for the production of textbooks in both the humanities and the sciences.

8.2 The Local Production of Schoolbooks

Searching for the extension of mathematics textbook production in Europe from 1775 to 1829, I undertook it to assess this production on the basis of an excellent bibliography of mathematical publications beginning in the 1500s (Rogg 1830). This bibliography seems to be quite reliable for publications in Central Europe, Germany, France, and England (and less so for peripheral countries such as

[2] Neohumanism is the expression, in the field of education and sciences, of the movement of profound reforms of the whole society in the state of Prussia after the defeat against Napoleon in 1806. A protagonist of neohumanism was Wilhelm von Humboldt. Neohumanism was not restricted – as is often reported – to promoting the humanities, but it was also important for the exact sciences.

[3] Es stimmt zwar mit den Grundsätzen einer vernünftigen Verwaltung des Schulwesens nicht überein, den Gebrauch bestimmter Lehrbücher in den Schulen im Allgemeinen vorzuschreiben, indem dies die Aufnahme besserer von Zeit zu Zeit erscheinender Lehrbücher, und somit die innere Schulverbesserung erschweren würde [...] Das unterzeichnete Departement enthält sich daher aller positiven Vorschriften bestimmter Lehrbücher in Schulen. [...] Indessen schließt diese Enthaltung von einem positiven Einwirken ein negatives Verfahren, das in Entfernung und Abhaltung des Schlechten besteht, nicht aus (quoted from Schubring 1991, p. 182).

Scandinavia, Spain, and Portugal, but also for Italy). Although it is natural that Germany is overrepresented in a bibliography published by a German researcher, its omissions regarding publications in other countries do not result in a qualitative difference in the result of their comparisons, as shown in the following three tables. I evaluated Rogg's bibliography in two dimensions:

- areas of mathematics, as identified by Rogg, according to the contemporaneous understanding of these areas;
- nationality of authors – based on the place of publication, but also on supplementary personal data. This was particularly important for Germany, where I intended to measure the level of mathematical activity in the various major States or groups of States. It proved efficient to aggregate the German data into four groups: Prussia, Bavaria, Austria (still part of Germany until 1866!) and the other German states.

Rogg started his bibliography from the invention of printing, but I preferred to start from 1775 to ensure comparability with the study by Jean Dhombres (1984), which extends from 1775 to 1825. I gathered the data in samples from consecutive 5-year periods. To limit the list to new publications, only the first edition of each textbook was included – with the exception of cases where the text has been radically changed or where a new or additional author appears.

- Table 8.1 shows the evaluation concerning geometry, assembling Rogg's entries for elementary geometry, analytic or higher geometry, trigonometry, and practical geometry.
- Table 8.2 provides the results for arithmetic, algebra, and calculus. For unknown reasons, Rogg's bibliography does not contain any ordinary French arithmetic textbook; the table shows two parallel items in each row, except for France, where the data refer to the number of algebra and calculus textbooks, while the italicised numbers represent the sum of arithmetic, algebra, and calculus textbooks. This measure aims to facilitate the comparison of French data with those obtained for the German States. A total of about 100 arithmetic textbooks seems to be a good estimate for France over the whole period.

Table 8.3 shows the totals from 1775 to 1829 for two particular mathematical subdisciplines: on the one hand, differential and integral calculus, and on the other hand, all mathematical subdisciplines in the various geographic regions.

Comparing these data confirms impressively how culturally context-dependent is mathematical activity: regarding textbook production, France was by no means dominant in Europe; its mathematical community being smaller than that of Germany. This difference becomes even more striking in a direct comparison between France and Prussia, when one takes into account the inverse relationship between the populations of these two nations: Prussia had no more than 13 million inhabitants in 1831, while France boasted 32 millions. Relations between German States are also very informative: Prussia is absolutely dominant, Bavaria comes in second, despite being smaller than Austria, the latter faring no better than the smaller German States.

France was not even a leader in calculus, either before or after 1800, despite the fact that the French are considered leaders in the development of this branch of mathematics during this period.

Table 8.1 Production of geometry schoolbooks

Years	France	Prussia	Bavaria	Austria	Other German states	Sum for Germany	Italy	Great Britain	Other European states
1775–79	4	2	4	1	6	13	1	1	2
1780–84	5	13	–	–	11	24	1	4	1
1785–89	2	4	3	2	7	16	1	5	2
1790–94	3	3	6	–	12	21	2	1	8
1795–99	5	14	3	–	28	45	–	2	3
1800–04	7	12	4	2	10	28	2	5	3
1805–09	9	14	5	1	17	37	–	–	2
1810–14	15	13	10	2	16	41	1	5	1
1815–19	8	19	8	1	29	57	1	1	3
1820–24	14	56	12	6	33	107	3	–	–
1825–29	21	44	3	6	34	97	–	2	1
s. d.	5	1						26	
Sum	**98**	**195**	**58**	**21**	**203**	**477**	**12**	**52**	**26**

Table 8.2 Production of schoolbooks in arithmetic (*Italics*), algebra/calculus

Years	France	Prussia	Bavaria	Austria	Other German states	Sum for Germany	Italy	Great Britain	Other European states
1775–79	4	*9* (2)	*7* (3)	*4* (2)	*20* (10)	*40* (17)	1	1	*6*
1780–84	6	*14* (7)	*4* (1)	*6* (3)	*19* (3)	*43* (14)	1	1	*2*
1785–89	2	*18* (11)	*3* (–)	*3* (2)	*38* (10)	*62* (23)	2	2	*3*
1790–94	3	*19* (9)	*5* (1)	*3* (1)	*35* (10)	*62* (21)	–	1	*7*
1795–99	11	*23* (12)	*10* (2)	*6* (1)	*48* (12)	*87* (27)	1	4	*4*
1800–04	6	*31* (9)	*9* (3)	*4* (–)	*49* (15)	*93* (27)	1	1	*2*
1805–09	7	*31* (10)	*7* (2)	*8* (4)	*52* (13)	*98* (29)	–	1	*4*
1810–14	8	*23* (6)	*8* (3)	*5* (2)	*58* (18)	*94* (29)	–	1	*7*
1815–19	1	*55* (22)	*17* (6)	*11* (5)	*73* (15)	*156* (48)	–	1	*2*
1820–24	8	*62* (18)	*24* (1)	*12* (3)	*77* (26)	*175* (48)	1	–	*1*
1825–29	12	*49* (20)	*13* (6)	*9* (2)	*54* (15)	*125* (43)	–	–	*6*
s.d.	3	*6* (2)	–	*4* (–)	*12* (–)	*22* (2)		*18* (9)	
Sum	**71**	***340* (129)**	***107* (28)**	***75* (27)**	***535* (147)**	**1.057 (331)**	**7**	***40* (22)**	***44***

Table 8.3 Overall production of schoolbooks 1775–1829

Years 1775–1829	France	Prussia	Bavaria	Austria	Other German states	Sum for Germany	Italy	Greta Britain	Other European states
Calculus	19	30	4	3	18	55	–	3	1
Analytic geometry	17	14	5	2	11	32	3	8	3
Sums Aritmética/Álgebra/Cálculo	71	340	107	75	535	1.057	7	40	44
Sum geometry	98	195	58	21	203	477	12	52	26
Sum for all mathematical areas	**169**	**535**	**165**	**96**	**738**	**1.534**	**19**	**92**	**70**

The subdivision into 5-year periods makes it possible to study changes in the production of mathematics textbooks: a peak is established from 1795 to 1799. Productivity growth in Prussia is particularly marked after 1819, during the period of stabilisation that followed. to the profound educational reforms of 1810.

Despite their praise of teachers' freedom to choose textbooks, Prussian school authorities were sometimes concerned about the steady publication of new books. But even when they wanted to intervene and take some action, they ended up confirming that the teachers were the ones who knew the best about teaching methods, and they really avoided interfering with their independence. In fact, almost every mathematics teacher seems to have struggled to produce his own textbook, as they all tended to be critical of the basic assumptions and procedures of available textbooks, which is a highly revealing expression of what I would like to call a "continuum" of epistemological positions, which is possible in Mathematics.

8.3 Crelle's Misunderstanding

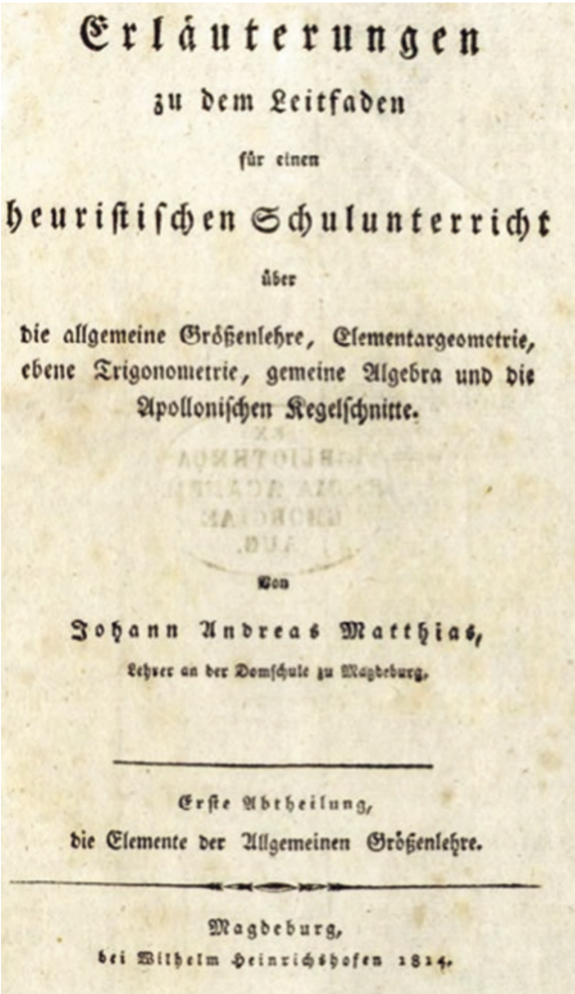
This very conception of teaching dominated by orality was at stake when August Leopold Crelle (1780–1855) was charged with consolidating and expanding the reform of mathematical education in Prussia as president of a committee which should develop such a plan in 1829. Crelle, a technician, self-taught in mathematics and then advisor for mathematics to the Prussian Ministry of Education, saw all the evils of contemporary mathematical instruction in the lack of homogeneity and proposed a centralist policy, following the French model: a uniform textbook should guarantee the homogeneity and high quality of the teaching. It took a long time for the other committee members to convince Crelle, at least partially, that centralising measures risked being counterproductive, and that oral tradition and the independence of teachers had to be maintained. Due to Crelle's dogged insistence on producing a model textbook, the only concrete decision made in a large number of meetings was to propose a concours to choose a new book. As a period of 2 years had been proposed for manuscripts to be submitted to the competition, this shows that just as in the case of the first competition in France, the real opportunities for change in the school were neglected in favour of the pursuit of some abstract ideal. In fact, no action was taken, and this chance to open approaches in favour of mathematics against the dominance of classical languages was lost (see Schubring 1988a).

8.4 Differentiation of the Schoolbook as a Kind of Publication

The fundamental change in the role of Gymnasium-teachers in Prussia is evidenced directly by a change of prototype and style in textbooks. At the same time, the new character of books confirms the changed function of orality.

In fact, the traditional type of "Handbuch" (compendium), voluminous and cumbersome, was replaced by a concise "Leitfaden" (guide), supplemented first by a methodological complement for use by the teacher, and later by an "Aufgabensammlung" (collection of exercises).

The precursor of such a modern schoolbook had been published and reprinted since 1813 by Johann Andreas Matthias (1761–1837), first a teacher of the Gymnasium, and then from 1814 director of the Domgymnasium in Magdeburg and even provincial school administrator. The term "Leitfaden" in the title is programmatic: in a radically concise way, all the subjects of school mathematics were exposed in just 160 pages! The supplementary innovation consisted of "Erläuterungen" (explanations): in three volumes, with a total of 706 pages: they constituted a genuine teacher's manual, full of comments and methodological suggestions (Fig. 8.1). This work can really be considered the first "German Didaktik and Methodik" for mathematical instruction.



Erläuterungen

zu dem Leitfaden

für einen

heuristischen Schulunterricht

über

die allgemeine Größenlehre, Elementargeometrie, ebene Trigonometrie, gemeine Algebra und die Apollonischen Kegelschnitte.

Von

Johann Andreas Matthias,

Lehrer an der Domschule zu Magdeburg.

Erste Abtheilung,

die Elemente der Allgemeinen Größenlehre.

Magdeburg,

bei Wilhelm Heinrichshofen 1814.

Fig. 8.1 Cover of the Guide for Teachers, by Matthias, first volume of 1814

Matthias was very explicit about his view of the relationship between teacher and schoolbook. As he explained in the first part of his “Erläuterungen”, the emphasis was on orality, where the “Leitfaden” was intended to support the oral part of teaching. Matthias highlighted the fundamental difference between a *Handbuch* and a *Leitfaden* in relation to the interaction between teacher and student: compendia, according to Matthias, are intended to present the subject to be taught to students in a closed and complete way, rather than enabling the students to discover by themselves – under the guidance of the teacher – what this subject is. The difference between a *Handbuch* and a *Leitfaden*, he added, corresponds to that between a lay teacher and a scientifically trained one: while the former is obliged to submit himself slavishly to an exhaustive treatise, being “always nothing but the executor [‘Organ’] of the textbook”, the scientifically trained teacher is able to use his own competence to guide and extend students’ reasoning, selecting subjects from the textbook to work methodically “in his own unlimited freedom” (quoted in Schubring 1988a, p. 11).

A schoolbook series of similar impact was published from 1820 onwards by Ernst Gottfried Fischer (1754–1831), mathematics teacher at a Berlin Gymnasium. His four-volume “Lehrbuch” was also supplemented by a guide for teachers, exposing on didactics and methodology of mathematics teaching, subdivided into four parts; these “Anmerkungen” (notes) occupy 560 pages. Although Fischer’s schoolbook for the students did not correspond exactly to the style of the “Leitfaden” – it comprised 1385 pages in its four volumes – this author soon had to surrender to the new format, publishing abridged versions, which were entitled “Auszüge” (excerpts), thus confirming the impact of the new style for elaborating textbooks of the latter type.

Another even more telling fact is that each of these pioneering teacher-directed schoolbooks experienced just one edition, while the corresponding books for students were reprinted many times. Nothing comparable to these first two methodological guidelines for teachers from the 1810s and 1820s was published for a long time.

An explanation can be found in the “Leitfaden” of G. Grabow (1793–1873), from 1823. Grabow, himself educated at the new University of Berlin in the spirit of Neohumanism, propagated that providing guidance for didactic issues was not admissible in manuals for teachers because such questions belonged to the teacher’s methodological autonomy. The teacher, he said, was better able to adapt the teaching subjects to the students’ cognitive capacities (cf. Schubring 1988a, p. 9).

The publication of textbooks with accompanying guidebooks for teachers and their use thus appear to have been only transitory during the first stage of neohumanist educational reform in Prussia, when mathematics had become one of the three main school disciplines, while teachers adequately trained were not yet available in sufficient numbers. As soon as the neohumanist ideal of scientifically prepared teachers was implemented for mathematics as well, teacher manuals lost their purpose. Under the mastery of oral teaching by a highly trained and qualified teacher, the concise “Leitfaden”, supplemented by a collection of exercises, has become the predominant textbook format. As the exercise collections could be used with any methodological conception, they became the real "bestsellers". Its main prototypes – Meier Hirsch, Heis, and Bardey – have seen numerous re-editions each.[4]

Even more striking was the shift towards a redefined orality in universities organised according to neohumanist principles. Excellent professors and researchers like Dirichlet and Weierstraß avoided writing textbooks. They presented the new knowledge in their lecture courses – and those who wanted to learn about the latest advances in research needed to come in person and attend the lectures. This is a dramatic difference compared with France, where higher education was tied to textbooks written by university professors, and students continued to use these officially recommended textbooks.

[4] The first to establish this new schoolbook pattern in Prussia was Meier Hirsch (1765–1851; with alternate spelling Meyer), of Jewish origin, with the collection *Beispielsammlung aus den Gebieten der Buchstabenrechnung und der Algebra*, published in 1804. There was a 20th edition in 1890 and even one in 1916. The second was Eduard Heis (1806–1877), who turned out to be the most successful author of an exercise collection: his *Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra: für Gymnasien, höhere Bürgerschulen und Gewerbschulen*, first published in 1837, experienced its 115th edition in 1910. The third major author was Ernst Bardey (1828–1897), with his *Methodisch geordnete Aufgabensammlung, mehr als 7000 Aufgaben enthaltend, über alle Theile der Elementar-Arithmetik*, first published in 1871. It not only was extended, to 8000 exercises, but also adapted to the various school types now becoming differentiated. During his lifetime, it reached its 23rd edition and was re-edited at least until 1940.

Chapter 9
Cultural Specificity of Textbooks: The Case of Legendre in Italy

9.1 The French Anti-euclidean Approach in Geometry and Legendre's Geometry

To a large extent, studying the peculiarity of mathematical communication in Italy, since its independence as a united national state (1861–1866), means to assess the relationship between Legendre's geometry textbook and Euclid's *Elements*.

To understand this probably surprising statement, I need to refer to the specific feature of French mathematics already mentioned in Chap. 4: France was the only European state that emancipated itself early, in Pre-Modern Times, by the mid-sixteenth century, from the use of Euclid's *Elements* as the standard mathematics textbook. The first to challenge Euclid's role was Petrus Ramus, who did not question one or another particular aspect or proposition, but rather attacked the declared prominence of Euclid's methodology, which was in fact the main justification for its use as a schoolbook, serving as the standard in so many countries for very long periods. Ramus denied the model character of Euclidean methodology, of its disposition of propositions and also of its systematics. To replace them, he developed a set of new methodological rules. The refusal of Euclid's role as a model gained more and more acceptance in France. Descartes' famous work *On Method* (1637) is based on Ramus' rules. And Arnauld's influential textbook *Nouveaux Élémens de Géométrie* (1667) reinforced Euclid's refutation and replacement by an alternative approach to geometry. In the eighteenth century, after all, this anti-Euclidean stance in France led, as a variant, to the publication of textbooks on geometry that renounced the demands for rigour and, on the contrary, started from alleged didactic needs of the learner. Typical of this new style of book, unknown in other cultures, is the textbook *Élémens de géométrie*, written by Clairaut and published in 1741 (see Sect. 4.4.).

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_9

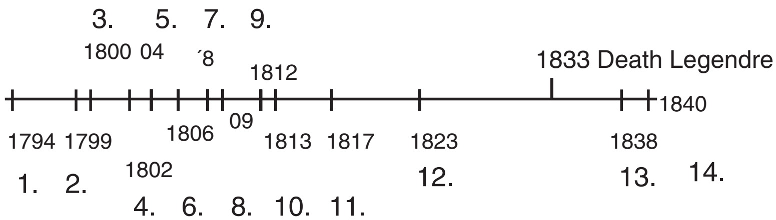
The French Revolution, with its programme of education for all, therefore meant a decisive break with this tradition; in order to implement the new communication system, it turned to the values of rigour. Indeed, Legendre's *Éléments de géométrie*, first published in 1794, are the first important result of this reorientation. In their report on the development of mathematics since 1789, published in 1810, Lacroix and Delambre emphasised the key role of this book:

> Mr. Legendre, in 1794, undertook it to revive among us the taste for rigorous demonstrations (Delambre 1810, p. 45).[1]

The composition of Legendre's book was motivated by two milestones in education reform: the creation of the first institution for training teachers at the level of higher education, the *École Normale* of the year III (1795), and the concours for *livres élémentaires* of 1794, involving all school subjects (primary). Organising in 1794 the *École Normale*, the *Comité d'Instruction Publique* had commissioned eminent scholars to write *livres élémentaires*; to compose arithmetic and geometry, Lagrange was appointed (on the 1st Brumaire, year III). Soon after (on 16 Brumaire, year III), Lagrange proposed to work together with Legendre. In fact, it was Legendre alone who took it upon himself to write the textbook, and he limited himself to geometry. Approximately 10 months later (3 Frutidor, year III), Legendre presented the printed text to the *Comité d'Instruction Publique*, who agreed to accept him as a competitor in the general concours that had been opened the previous year, on 9 Pluviôse, year II. Although the competition had asked for manuscripts, the already printed *Éléments* de Legendre won a jury prize and the best evaluation in mathematics (see Sect. 6.2.).

Legendre's geometry book experienced enormous success; in France, it was many times re-edited. During Legendre's lifetime, there were 12 editions – he used to rework on new editions. After Legendre's death, in 1833, there was a thirteenth and a fourteenth edition, until 1840 (Table 9.1), when its publisher, Firmin-Didot, decided to charge Marie-Alphonse Blanchet (1813–1894) to revise Legendre's master piece, even though he was a mathematics teacher who had never before showed any qualification for doing so. His revision resulted in profound conceptual changes (see Schubring 2007, pp. 44 ff.). Already the title indicated changes: "New edition, with additions and modifications". And the preface attributed "imperfections and some gaps" to Legendre's original version.

Table 9.1 The original editions



[1] M. Legendre, en 1794, entreprit de faire revivre parmi nous le goût des démonstrations rigoureuses.

Right from the beginning, one remarks on a deviant conception. Legendre started from his general description of what geometry is: a science that aims to measure extensions. Next, Legendre introduced the three dimensions of extension: length, width, and height, in order to characterise the three basic concepts of geometry: line (length without width); surface (having length and breadth, but no height); solid or body (having the three dimensions of extension). By contrast, Blanchet started from the notion of body and volume and deduced the notion of surface as the limit that separates the body from the surrounding space, and the line as the meeting of the surfaces of two bodies. The point is the place where two lines intersect (Blanchet 1845, p. 1).

The use of the term "limit" was not accidental. In fact, already in the preface, Blanchet mentioned as a decisive change to substitute, in demonstrations on circles and round bodies, the demonstration method by the absurd with demonstrations "by the method of limits", even admitting that this method belongs to the field of higher mathematics (ibid., ii). Evidently, using a key method of analysis was totally alien to the concepts of Legendre's geometry! And indeed, Blanchet was obliged to give, within geometry, a brief course on what would be the method of limits (ibid., pp. 115–121). And as for the theory of parallels, one of the subjects of major care and attention for Legendre, developed in extensive notes in addition to the main text, Blanchet returned to the simple postulate, in the form of the theorem of the sum of angles in a triangle, "proved" without any qualms. Blanchet had even neglected Legendre's careful differentiation in proofs between commensurable and incommensurable cases (ibid., p. 60). In later editions, Blanchet continued to further revise the already mutilated text and to add appendices to it – and this process of transforming the text was continued, referring to new mathematics curricula prescribed by the French ministry.

While this interference in Legendre's almost classical textbook is already quite unique, there is a second likewise unique connected fact. The publication of Blanchet's first revision occurred in an extraordinary and, to my knowledge, unprecedented way in the history of textbooks: it was a double book! The revised text was followed by Legendre's original text, each totalling 285 and 271 pages respectively. Blanchet explained that he initially only intended to insert his modifications into the original text but later decided, in particular due to parts where he reversed the order of the propositions, to publish his own version as a complete and coherent work followed by the "old text of Legendre", affirming that such a complete work would be of interest to the students. Thus, "the new edition is also convenient for people who adopt Legendre's treatise without any change" (Blanchet 1845, ii). One understands the publisher's reason for the double textbook when regarding the issue of printing the diagrams. It proves that Blanchet used the same diagrams and the same "planches", because the elaboration of diagrams by copper engravings constituted a decisive economic factor in book printing.

On the other hand, Firmin Didot, in 1845, already disposed of a new printing technique for diagrams, enabling the inserting of good quality diagrams within the text. Thus, in this first edition, the text referred, in the margin, to the number of diagrams in the tables, at the end of the double volume, and sometimes to diagrams printed within the same page. The duplication of the two versions was maintained in the second edition, in 1849; however, since the third edition, in 1854, Firmin Didot published Blanchet's version without the "appendix" – they had then succeeded to transform all original copper engravings into diagrams printed within the text, thus without the final tables.

Blanchet's version ran from its first edition, in 1845, to the 37th, in 1894 (Table 9.2). Legendre's original (12th) edition continued to be published; however, during the second half of the nineteenth century, in France – by the same publisher, Firmin-Didot, and with the original copper engravings printed at the end – and in pirate editions in Belgium.

Table 9.2 The Blanchet editions in France

1845	52	57	60	64	72	75	81	89	1894
1.	2.	3.	8.	11.	15.	19.	26.	30.	37.

Besides being a dominant geometry textbook in France throughout the nineteenth century, it also experienced an enormous and unprecedented international success. Crelle, for instance, had translated its 11th edition (1817) in 1822, praising its qualities in his preface: the book "is distinguished by the richness of content, by the clarity, order, and consistency of the exposition, by the accuracy and rigour of the demonstrations" (Crelle 1822, p. iii).

I have found at least 16 translations to other languages – in Europe, Asia, and North and South America. Legendre's book can, therefore, be assessed as the most extensively internationally disseminated modern schoolbook. The following list shows the countries and languages so far identified with the first respective translation (Table 9.3):

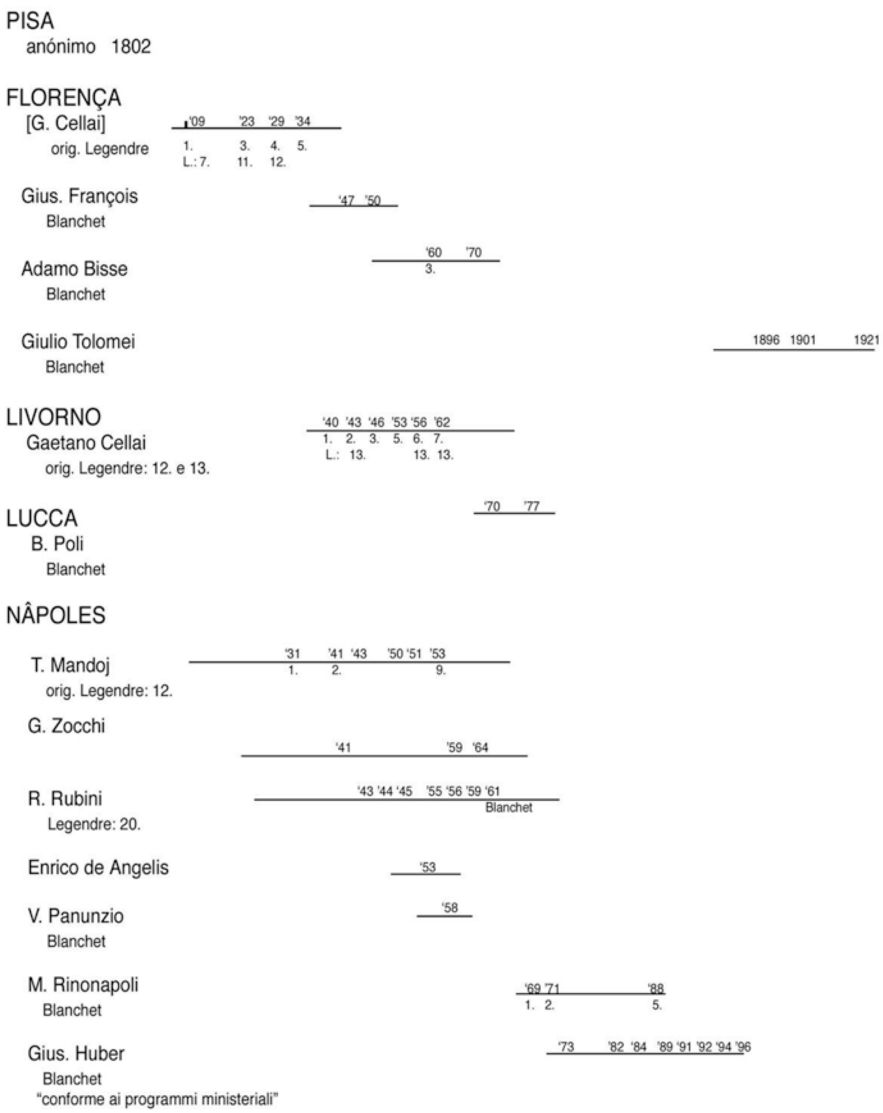
Table 9.3 The first translations in various countries

1802 Italy
1807 Spain
1809 Brazil
[1810/1812 Greece]
1819 United States of America
1819 Russia
1822 England
1822 Germany
1825 Sweden
1829 The Netherlands
Ca. 1830 Switzerland
1836 Ottoman Empire
1856 Persia
1866 Columbia
1876 Venezuela
1881 Egypt

9.2 The Decision in Italy for Euclid

As a matter of fact, Italy presents the most contradictory case among these international receptions of Legendre's geometry. On the one hand, Italy was the first country where a translation was published, in 1802 – actually in Pisa, then part of the French-influenced satellite-State Kingdom of Etruria. On the other hand, in Italy, the greatest number of independent translations were published, throughout the nineteenth century (see Table 9.4). This multiplicity seems to be due, in the first half of the nineteenth century, to the multitude of Italian States and territories before 1860, and in the second half of

Table 9.4 Italian translations of Legendre's geometry, from: Schubring (2009, p. 372)



the nineteenth century, despite the official ban of Legendre, to the competition between the original version and the Blanchet adulterated version, and moreover to an apparent independence of the former Kingdom of the two Sicily's, with Naples as capital.

A particularly revealing event for the issue of *communication* between different national communities was the decision of 1867 defining the mathematics curriculum for the secondary schools in the now unified Italy: Legendre's book was accused of lack of rigour and it was decreed to use Euclid's *Elements* as schoolbook, praised for its rigour (Vita 1986). The decision in favour of Euclid at such a late period has often attracted the attention of Italians and foreigners; however, the interest has been due more to curiosity, and the decision had not been really analysed. Historical accounts were restricted to Italian authors who largely shared the underlying meanings of this decision, so that the epistemological issues involved had not yet been discussed. The assessment by Vincenzo Vita, who had published a history of the Italian mathematics curriculum, was characteristic. He had listed the following criticism:

- Legendre treated the theory of parallel lines differently from Euclid;
- he had omitted Euclid's book V;
- he had converted the proportions between quantities into ratios of numbers;
- and in general, he had made a frequent use of algebraic methods, without "excessive" care for rigour (Vita 1986, p. 5).

A re-assessment therefore requires a comparative approach from an external andnot Italy-internal standpoint.

The first major reason for introducing Euclid's *Elements* was clearly political: the educational system was also subject to the nationalism that had inspired the unification movement. As Luigi Cremona (1830–1903), along with Giuseppe Battaglini (1826–1894), a member of the commission that drafted the new programme in 1867, had pointed out in 1860, the liberation of the "giogo straniero" (foreign yoke) also meant the liberation of "the most infamous books of Moznik [sic!], etc. [...] *We* do not have good elementary books that are Italian originals and reach the level of today's advances in science" (cited in Cremona 1914, p. 207). In fact, Močnik's textbooks, published since 1851, were the most used mathematics textbooks in Austrian secondary schools, and likewise in the central and eastern parts of Northern Italy. And the struggle against Austria, which had had Lombardy and Veneto under its control since 1815 and which played a role of political domination in Italy as a whole, was the main focus of the unification movement. On the other hand, in Northwest Italy (Piedmont and Savoy), translations of French books were used and here Legendre's book was the best known: (see the cover of the 1802 translation).

ELEMENTI

DI

GEOMETRIA

DI

ADRIANO M. LE GENDRE

PER LA PRIMA VOLTA

TRADOTTI IN ITALIANO

Si quid novisti rectius istis,
Candidus imperti.

PISA

DALLA TIPOGRAFIA

DELLA SOCIETÀ LETT.

MDCCCII

Cover of the first of the multiple regional Italian translations of Legendre's geometry, of 1802.

The second main reason was the intention to achieve an adequate integration of mathematics teaching into the dominant values of Italian secondary schools. Such values were defined by literary studies and classical languages (cf. Scarpis 1911, p. 27). In the commentary for teachers on the 1867 curriculum, the notion of the usefulness and applicability of mathematical knowledge was denied, and replaced by its function of "mental gymnastics to develop reasoning skills" (ibid., p. 26). Enrico Betti and Francesco Brioschi, the editors of Euclid's *Elements* (Betti & Brioschi 1867), now decreed for Italian schools (theirs followed Viviani's edition in 1690), in compliance with the curriculum, in their preface, emphasised the common function of the classical languages and of mathematics to serve as "intellectual gymnastics" (Betti and Brioschi 1868, p. v). To conform to such a legitimising function, the "harmful confusion" with practical or professional aims of mathematical teaching should be eliminated, and mathematics had to be "coordinated with the system of classical studies and defined in such a way as to form an integral part of a common instruction" (Betti and Brioschi 1868, p. iv). Apparently, the classical values were much stronger than in any other European country, as the "coordination" with this system resulted in a degree of commitment to the purity of method that surpassed the determinations of school mathematics in other European countries.

Table 9.5 From a manuscript version of the later shortened published version of (Giacardi and Scoth 2014), with kind permission of the authors

	Ginnasio					*Liceo*		
Year	1	2	3	4	5	I	II	III
Hours	0	0	0	0	5[a]	6[b]	7½[c]	0

[a]Arithmetic – Geometry (Euclid, Book I)
[b]Arithmetic – Algebra – Geometry (Euclid, Books II and III)
[c]Algebra – Trigonometry – Geometry (Euclid, Books IV, V, VI, XI, and XII, with the addition of the presentation of the circle, cylinder, cone and sphere according to Archimedes)

In fact, the conviction that only the classical text by Euclid would be capable to initiate students to rigorous mathematical thinking and the intention to integrate mathematics teaching optimally into an overall classicist curricular conception, led leading Italian mathematicians members of the curriculum committee to have established a likewise extreme and unique allocation of weekly teaching hours for mathematics over the eight grades from the *ginnasio* to the *liceo*. During the first 4 years, there would be no mathematics teaching at all (Table 9.5)! Students should be more mature to be able to learn from Euclid's *Elements* – and this should be possible from the fifth year on, the last year of the *ginnasio*. Yet, after some concentration of weekly hours over 3 years, the last year would again be without any mathematics.


GLI

ELEMENTI D'EUCLIDE

CON NOTE,
AGGIUNTE ED ESERCIZI

AD USO DE' GINNASI E DE' LICEI

ENRICO BETTI E FRANCESCO BRIOSCHI.

FIRENZE.
SUCCESSORI LE MONNIER.

1867.

Cover of the Euclid edition published by Betti and Brioschi in 1867

This striving for *purity* leads to the third main reason for the unanimous and absolute rejection of Legendre's approach to geometry. In the preface by Betti and Brioschi, the primordial polemic is directed against Legendre: it is affirmed that Euclidean geometry constitutes a complete science that is self-sufficient and that does not need, in any of its demonstrations, the support of the science of numbers (ibid., pp. vi–vii). Indeed, the underlying epistemological issue was the relationship between geometry and arithmetic/algebra. Legendre was "accused" of having mixed both branches in his geometry, making the book unsuitable for the intended methodological instruction. Legendre had, in fact, emphasised to use algebraic means when facilitating:

> The Ancients, who did not know algebra, remedied this by the reasoning with and by the use of proportions, which they handled with great dexterity. For us, who have this instrument more than they, we would be wrong not to use it if it can result in greater ease. I therefore did not hesitate to employ signs and operations of algebra, when I judged it necessary: but I was careful not to complicate with difficult operations what must be simple by its nature; and all the use I have made of algebra in these Elements is reduced, as I have already said, to a few very simple rules which one can know almost without suspecting that it is algebra (Legendre 1794, p. vij).[2]

In all Italian reflections of this period, the praise of the educational function attributed to Greek geometry is accompanied by the polemic against the "mixing" of geometry with arithmetic and algebra. Although recommending the Euclidean method as the most suitable for creating the ability of rigorous reasoning in students, the commentary to teachers, in 1867, warned against "clouding the purity of the geometry of the ancients by the transformation of geometric theorems into algebraic formulas" (cited in Vita 1986, p. 7). It is very characteristic of the underlying mathematical epistemology that geometry was conceived exactly as in the original Greek terms of proportions, so that no modernisation in the sense of introducing numbers was allowed, and arithmetic remained rigidly separate from geometry. The instructions for the teachers alerted them to avoid "replacing the concrete quantities (lines, angles, surfaces, volumes) by their measures", while encouraging "to always reason the first ones, even where the ratios are considered" (ibid.).[3]

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[2] Les anciens, qui ne connoissoient pas l'algèbre, y suppléoient par le raisonnement et par l'usage des proportions, qu'ils manioient avec beaucop de dextérité. Pour nous, qui avons cet instrument de plus qu'eux, nous aurions tort de n'en pas faire usage s'il en peut résulter une plus grande facilité. Je n'ai donc pas hésité à employer des signes et des opérations d'algèbre, lorsque je l'ai jugé nécessaire: mais je n'ai eu garde de compliquer d'opérations difficiles ce qui doit être simple par sa nature; et tout l'usage que j'ai fait de l'algèbre dans ces Eléments se réduit, comme je l'ai déjà dit, à quelques règles très-simples qu'on peut savoir Presque sans se douter que ce soit de l'algèbre.

[3] sostituire alle grandezze concrete (linee, angoli, superfici, volumi) le loro misure, [while encouraging] a raggionare sempre sulle prime, anche là dove se ne considerano i rapport. The curriculum of 1867, with the commentaries, are published in: Decreto Coppino (10.10.1867), Supplemento alla Gazzetta Ufficiale del Regno d'Italia, Firenze, 24 ottobre 1867.

9.3 Legendre's Critics in Italy

All Italian commentators agreed that Legendre's book violated this epistemology - his active opponents Cremona, Betti, and Brioschi, as well as their contemporaries, but also later authors of the history of mathematical teaching such as Scarpis (1911) and Vita (1986). Even Gino Loria, one of the few Italian mathematicians supporting curricular reforms, criticised in 1904 Legendre's lack of rigour due to his "méthode demi-arithmétique" (Loria 1905, p. 595). Vita's opinion, already mentioned, was typical (Vita 1986, p. 5).

How should one evaluate such criticisms? Certainly, an internal discussion would seem possible to some extent. Thus, one might think that one could have argued with the Italians about the advisability – in a mathematical and pedagogical sense – of maintaining a purely geometric doctrine of proportions, and of separating them from their measures, that is, from the numbers. Even more so, since Betti and Brioschi themselves felt obliged, in their 1867/1868 edition of Euclid, to present modern notions of geometry in an appendix. As an essential key to the passage to a modern understanding, they discussed there the relationship between proportions and their measures by ratios and how to deal with incommensurables.[4] Their justification for adding such a modernising appendix was didactic: to save time in teaching and facilitate learning (Betti and Brioschi 1868, p. 387). A discussion and negotiation with Italian critics on such didactic issues, however, would not be fruitful: brevity, ease of learning, and modernity were not pedagogical values culturally shared then in Italy. The most basic issues concerned favouring "pure" geometry, and these were epistemological issues where right or wrong were not a matter of debate, but where the positions maintained must be accepted as culturally legitimate conceptions of mathematics and its architecture.

In fact, Legendre had composed his geometry textbook by striving for an entirely different vision of rigour: the mutual support of algebra and geometry. It was based on this objective of generalising the functions of mathematics that Legendre had explained in his book III, when discussing the proportions of figures, that several of his demonstrations used algebraic devices and that he had recommended to "thus interweave the study of the two sciences": algebra and geometry (Legendre 1794, p. 59).[5] Crelle had endorsed this epistemological standpoint in his translation via supplementary notes, indicating that the amount of purely geometric knowledge that can be presented without the means of arithmetic and algebra is very restricted and was in fact given by Legendre in his first two books; and Crelle pointed out that the use of the "art of calculus" could not be avoided in geometry and was even necessary as soon as one progressed to the study of the magnitude and similarity of

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[4] A highly informative presentation of the historical and conceptual evolution of the theory of proportions and its relationship with numbers is made in Rouche's book (1992), in its first part *Des grandeurs aux mesures* ("From magnitudes to measures").

[5] entremêler ainsi l'étude des deux sciences.

figures. The use of arithmetic and algebra, Crelle explained, was not detrimental to rigour in geometry – on the contrary, it increased its rigour, since these disciplines are more abstract. Crelle concluded that its use in geometry should not be minimised, but maximised (Crelle 1833, p. 65).[6]

Thus, the curricular decision of 1867 in Italy represents for the historian one of the notable cases of pre-eminence of epistemological choices for mathematical practice.

9.4 Criticism in Gemany

Furthermore, the prosecutors from France and Italy were not aware that there was also in England and Germany a context of critical discussion of Legendre's approach: the imputation that "Legendre (Éléments de Géom. II. Note) wanted to deduce the doctrine of similarity from the already presupposed applicability of algebra to geometry" is presented in Hermann Hankel's (1839–1873) famous book on complex numbers (1867) (Hankel 1867, p. 66).

This general accusation can only be understood somewhat better by means of an analogous, but more extensive criticism in Möbius's book on the barycentric calculus (1827) to which Hankel referred (as well as to a commentary by Brewster in his 1822 translation of Legendre). Möbius postulated the need for a demonstration that what can be determined by construction can also be deduced by calculation. And Möbius accused Legendre of having presupposed the doctrine of similarity without having demonstrated the applicability of analysis to geometry (Möbius 1827, p. 190). But Möbius's text has the advantage of exposing the case to what Legendre was accused of: as already shown by his reference "II. Note", as mentioned by Hankel, this was not a part of Legendre's general exposition of geometry. Indeed, in the general text, the concept of similarity was introduced in Book III, as a consequence of his introduction of proportions as ratios between numbers.

[6] Betti, Brioschi, and Cremona had been encouraged by a similar attack on Legendre's geometry undertaken in France by Jules Hoüel, and it is curious to note that both fundamentalist returns to Euclid's original had also been inspired by the attraction that non-Euclidean geometry had represented for Cremona and Houël. Jules Houël (1823–1886), professor of mathematics at the *Faculté des Sci*ences in Bordeaux, was – after nearly three centuries! – an early and truly lone advocate of using Euclid's *Elements* as a model for schoolbooks in France. He accused Legendre, in the same vein as his Italian colleagues, of having substituted *grandeurs géométriques* with numbers, and of mixing "the geometric procedures of modern analysis" with geometry. On the other hand, Houël was critical of Legendre's use of methods borrowed from Greek geometers – in order to extend propositions established for proportions involving rational ones to incommensurable ones (Houël 1867, pp. 3–4; see Legendre 1823, pp. 45–47). In their "modern" appendix, Betti and Brioschi had contented themselves with a brief argument about limits for extending all propositions from commensurable to incommensurable quantities (Betti and Brioschi 1867/1868, pp. 388–389).

Indeed, *Note II* does not introduce the concept of similarity but discusses the problem of proving the parallel postulate. And since the twelfth edition, in 1823, he had presented an additional “demonstration”, using the so-called law of homogeneity. Applying this law, first introduced by Lambert, which says, if an angle in a triangle is a function of the two other angles and of one side, the angle must depend only on the two angles because they can be expressed by absolute numbers, and no longer on the side because this depends on an arbitrary unit of length (Legendre 1823, p. 281). Although well accepted in France at the same time, this demonstration of the postulate was later revised by Crelle in his 1833 translation, thanks to the additional presupposition of a law of homogeneity. But Legendre’s approach had the merit of demonstrating the difference between angles, which have an absolute measure, and lengths, which do not. This difference was one of the elements in the birth of non-Euclidean geometry.

Thus, in his *Note II*, Legendre did not intend to have an “analytical establishment of the doctrine of similarity” in order to treat lengths by means of algebra. He tried to apply the so-called law of homogeneity – and this issue had not been discussed or criticised by Möbius and Hankel. Indeed, for Legendre it became clear that the ratio between two lengths constitutes an algebraic quotient.

9.5 The Analytic Against the Synthetic

The role of Euclid’s book V is thus, in fact, revealing. One might think that it would have been possible to discuss with the Italians the suitability of book V for schools: it is one of the most complicated parts of the *Elements* and focuses - by discussing of the notion of incommensurability – on the theory of real numbers, thus transcending geometry (cf. Dhombres 1978, chap. 1). However, real numbers were not a topic of interest to Italian mathematicians in that period. They continued to understand the theory of proportions as constitutive for geometry. Even later, when the ministry had relaxed some of the original rigidity and allowed the use of a modern author for spatial geometry - instead of Euclid’s books XI and XII – the dominant character of books I to VI was maintained (cf. Scarpis 1911, pp. 26–27).

Cremona, in his own research as a specialist in geometry,[7] adhered to geometric fundamentalism. In a letter to *Giornale di Matematiche*, written in 1869 together with Brioschi, to refute criticisms of the 1867 decision, he showed himself to be an advocate of purism. While even granting that there are defects in Euclid’s *Elements*, they argue that Euclid should be revised according to his own spirit, “provided it be

[7] Before being appointed, in 1860, to the chair of higher geometry at the University of Bologna, Cremona had worked as a professor of mathematics in a *ginnasio* (lower grades of secondary school).

guaranteed that true geometry will be practiced and in no way arithmetic" (Brioschi and Cremona 1869, p. 54).[8]

This pure, fundamentalist geometry was complemented by a second theme that constitutes another peculiarity of the Italian value of school knowledge: "Aritmetica razionale" (rational arithmetic). Unfortunately, this appears to be a hermetic Italian discipline, being an element of the curriculum; Italians did not need to explain their essence to each other, and they were never questioned by foreigners. An *Aritmetica razionale* is opposed to practical Arithmetic (as taught by the Austrians!) and intends to present arithmetic as an axiomatic theory, in a deductive way and by the use of proofs, focusing mainly on the natural numbers (Menghini 2012).

In fact, such conceptions of school mathematics were developed by university mathematicians. No genuine group of mathematics teachers who could have demanded that the curriculum be organised in a didactically bottom-up fashion seems to have existed.[9] There were several attempts to replace deductive instruction with intuitive and applied approaches, but all failed after a short time and there was always a return to deductive teaching (cf. Scarpis 1911; Vita 1986). Heated debates and quarrels took place due to another "anti" –Euclidean reform movement: that of the "fusionists", who preached since the 1880s – opposing the "separationists" – a fusion between plane and solid geometry. The admission of fusionist textbooks in the early twentieth century represented important progress (Menghini 2019).

9.6 Cultural Consequences of the Classicist Conception

However, all efforts by the early twentieth century to stabilise mathematics by modelling it according to the notions of classical education, without appropriate didactic conceptions, proved to be counterproductive: the status of mathematics as a school subject gradually deteriorated. The disastrous results of the (centrally organised) school graduation exams in 1878 were decisive: written exams became optional.

[8] Remarkably enough, textbooks for Italian secondary schools still continue to emphasise their adherence to scientific rigour in geometry, to applying Euclid's model to give the teaching of mathematics a formative character. There are many textbooks that are either entitled "geometria razionale" or that claim to present geometry in a "rational" form, that is, as a deductive theory. The presentation usually starts with some elements of logic and set theory, and applies them to the fundamentals of geometry. By default, there are chapters on the theory of proportions and the measurement of quantities, followed by some notions about real numbers (cf. Palatini and Dodero, *Corso di Geometria*, 1992; Conti and Baroncini, *Geometria razionale per i licei*, 1991; Palatini and Faggioli, *Elementi di Geometria per i licei classici*, 1989; Franzetti & Nicosia, *Matematica, Linguaggi e teorie*, 1990).

[9] One of the reasons for missing the professional activity of mathematics teachers might be the fact that secondary schools were secularised in Italy only in 1867.

It must also be considered that, in the school structure of 1867, mathematical instruction had been restricted to three upper grades. When mathematics was later introduced into the lower grades of the *ginnasio*, practical arithmetic was admitted there to be taught.

After all, by 1904, mathematics had been reduced to an elective subject in the upper grades of the school system, limited to a minor subject (Scarpis 1911, pp. 27–31). In fact, Italy was the only European country where mathematics lost its status as a major subject in school.

The function and structure of mathematical studies in the universities of Italy were also peculiar. The first 2 years were for common studies for future teachers (and scientists) and engineers. After an examination, the groups split up: the majority continued on to application schools (mathematics, therefore, had a polytechnic function for these), while a small percentage (less than 10%) continued mathematical studies – mainly the women, as Pincherle, were dissatisfied and complained (Pincherle 1911, p. 5). The purely geometric spirit of school mathematics certainly constitutes a decisive reason for the remarkable change in Italian mathematical research: although algebraic research dominated in the first half of the nineteenth century, it was completely supplanted by geometry and by research on logical foundations at the end of that century (cf. Gispert 1984).[10]

[10] The discrepancy between this state of mathematical instruction and the number of important mathematical researchers in the Italian Risorgimento might cause admiration. However, there is no immediate impact of the quality of mathematics instruction in a given country upon the productivity of top mathematicians. Rather, such a quality affects the extent of mathematical culture and its social legitimacy. In fact, four of the most famous Italian mathematicians were already active before 1860: Beltrami, Betti, Brioschi, and Cremona - thus being the situation comparable to that of France, where the eminent mathematicians who educated the prolific research generation emerged before the revolutionary period (Lagrange, Laplace, Monge, etc.), when the state of mathematical instruction was poor, too. Also comparable to the Paris school *École Normale Supérieure*, the *Scuola Normale Superiore* in Pisa continuously produced mathematical researchers only during a certain period (1867–1877), and this occurred much more rarely in the decades that followed. In fact, mathematical culture was not sufficiently “rooted” in Italian society to be able to prevent the degradation of mathematics as a school discipline.

Chapter 10
Transmission of Textbooks from Metropoles: The Nineteenth Century

The establishment of public-school systems, in the aftermath of the French Revolution, turned out to unravel qualitatively new developments in the history of mathematics textbooks. So far, we have analysed three countries with particularly significant patterns of these developments in the new era: France, Prussia/Germany and Italy. We will now try to report what research has shown so far about textbook history in other European countries, and also in non-European regions. It turns out that one can understand the first periods in these other countries largely as a transmission from metropoles – actually, from two: France and Prussia resp. Germany – to the "peripheries" of these centres of mathematics education. Italy remained a rather isolated case, with some analogous features revealed in England. France was disseminated and received, with its mathematics textbooks, in the broadest manner – also outside Europe and even in Islamic countries. But there were also countries with a rather independent development, based on a longer proper tradition; one such case is The Netherlands.

10.1 France

Let us first discuss the schoolbook situation in France during the nineteenth century. Due to the centralist educational policy, maintained also by the governments after the fall of the Napoleonic Empire, the number of published textbooks continued to be rather small, and written by few authors, dominantly from higher degrees of the social hierarchy and seldomly written by mathematics teachers. In the first half of the nineteenth century, the three main authors – Bézout, Lacroix, and Legendre – were joined in particular by Pierre-Louis Bourdon (1779–1854), in the 1820s. Bourdon, a school inspector for secondary schools, published a rather complete set for school mathematics: an *arithmétique*, an *algèbre*, an *application de l'algèbre à*

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
https://doi.org/10.1007/978-3-031-17670-8_10

la géométrie, a *trigonométrie* and co-authored a *géométrie* schoolbook. His series was quite successful; most of the books gained adapted versions after his death, until the twentieth century.

A telling example for the dominance by higher education is the *Cours complet des mathématiques élémentaires*, prepared by the publisher Armand Colin, in Paris, 1894. Its quality was signalled by being written "sous la direction de M. Darboux". Gaston Darboux (1842–1917) was one of the leading French mathematicians and professor at Sorbonne. The first four volumes were on *Arithmétique théorique et pratique*, *géométrie élémentaire*, *algèbre élémentaire,* and *trigonométrie rectiligne,* with all authors being mathematicians from higher education: Jules Tannery (1848–1910), Carlo Bourlet (1866–1913), and Jacques Hadamard (1865–1963). Other eminent mathematicians were competitors, for instance, an *arithmétique* by Joseph Bertrand (1822–1900), and an *algèbre* and a *géométrie* by Émile Borel (1871–1956).[1]

The most immediate transmission of French textbooks occurred in Spain, Portugal/Brazil, and in the United States. Later, strong transmissions occurred in Latin America.

10.2 Spain

Spain was the second country to publish a translation of Legendre's geometry, in 1807. In the first decades of the nineteenth century, Spanish schoolbooks for mathematics were basically translations of French ones; besides Legendre and Lacroix, it was Bourdon, for his arithmetic and algebra. Moreover, there was Louis-Benjamin Francœur (1773–1849), professor at *Faculté des sciences,* in Paris, who had in France first been prescribed for his mechanics textbook, but his *Cours complet de Mathématiques pures* came into use here. For higher parts of mathematics, it was likewise again translations from France: differential and integral calculus by Jean-Louis Boucharlat (1775–1848) and Claude Navier (1785–1836), professor at *École nationale des ponts et chaussées*; mechanics by Siméon-Denis Poisson (1781–1840), professor at the École Polytechnique, and descriptive geometry: by Théodore Olivier (1793–1853), professor at the *Conservatoire National des Arts et Metiers* (Ausejo 2014, pp. 287 f.).

In Spain, establishing secondary schools constituted a complicated process throughout the first half of the nineteenth century. Between 1846 and 1852, the government published lists of admitted schoolbooks for the secondary schools. In the first of these lists, of 1846, the prescribed ones were those by three French authors: again, by Bourdon, Lacroix, and Legendre (Ausejo 2014, p. 288).

[1] A list of the textbooks for the various parts of school mathematics used in 1910–1911 is published in: CIEM 1911, pp. 151 ff.

10.3 Portugal/Brazil

Portugal presents the most striking example of a direct transmission of French textbooks. Given the imminent occupation of the country and its capital Lisbon by French troops in 1808, not only the royal family and its court fled to Brazil, governing now the remaining territory and colonies from Rio de Janeiro as the capital, but also Academy members and an enormous number of technical and scientific equipment were transferred – including libraries. Moreover, while over centuries Portugal had maintained Brazil in an underdeveloped state, having to serve as a provider of colonial products, now – eventually – a printing press had been transmitted, too, within this fleet.

Creating educational institutions, one of the first was in higher education, in 1810, the *Academia Militar*, for the formation of military and civil engineers. Inspired by the *École Polytechnique* in Paris, mathematics became the basic preparatory course. The royal founding document for the engineering school listed in detail all the textbooks which should be used in the school's courses. The surprising fact was that, although the getaway from Portugal and taking residence overseas was to escape from French troops, all textbooks for mathematics were recent ones by French authors, with the only exception of Euler's algebra, which should be used additionally to Lacroix's algebra. The main textbooks were:

- Lacroix, arithmetic,
- Lacroix, algebra,
- Euler, algebra,
- Legendre, geometry
- Legendre, trigonometry[2]
- Lacroix, application of algebra to geometry
- Lacroix, elementary treatise of differential and integral calculus,
- Monge, descriptive geometry
- Francœur, mechanics (Saraiva 2007).

All these books should be translated into Portuguese, and, in fact, an intense activity of translating was initiated. The first three translations were published already in 1809: Euler's algebra and Legendre's geometry and trigonometry. The other Portuguese translations were published between 1812 and 1814 (Saraiva 2014, p. 97).

Regarding textbooks for teaching the differential and integral calculus in Brazil, not only all the textbooks used there in teaching were either French or French-inspired ones, but the ones used for training military engineers in the nineteenth century were mostly French textbooks – 32 – besides only one by British Augustus de Morgan, as shown by data gathered on calculus textbooks in stock in the great library of *Academia Militar das Agulhas Negras* (Rezende, state of Rio de Janeiro), one of the follow-on institutions of the *Academia Militar* (Pereira 2017, pp. 26 f.).

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[2] Legendre had inserted an additional part on trigonometry since the second edition of his geometry. This additional part has in many foreign translations been published as a separate textbook.

10.4 Latin America

For the nineteenth century, one remarks the analogous tendency in other Latin-American countries, in the earlier Spanish colonies. Colombia and Venezuela are the most thoroughly studied. For these two countries, Carlos Oliveira has shown, in his master's dissertation, the particular pattern of a double transmission. While there were used textbooks, translated and adapted directly from the French originals, there were others which revealed a transmission first from France to Spain, and then from Spain, again adapted to a Latin-American country. Lacroix's schoolbooks were in use in Colombia and in Venezuela in French versions, in translations published in Spain, and in new translations published in Venezuela (Oliveira and Schubring 2021, pp. 92 ff.). And Legendre's geometry showed even more revealing features: after that at first the Spanish translation of an original edition had been used, the translation published in Venezuela in 1879 was one of its original French versions, while the version translated in Colombia in 1866 was one of Blanchet's modified editions (ibid., pp. 121 ff.).

In particular, descriptive geometry was a teaching subject where one used, in Latin America, either a French original textbook or a translation. In Chile, for instance, a translation of C.F.A. Leroy's book was used. And in Colombia, when a *Colégio Militar* was opened in 1848, enabling studies of higher mathematics, a great number of French textbooks were bought, including the books by Gaspard Monge and by Louis-Léger Vallée for descriptive geometry (Schubring et al. 2019, p. 378 f.).

10.5 United States

The first institution in the United States enabling studies of higher mathematics was the United States Military Academy at West Point, founded in 1802. The title of a book about its history already indicates this important role: *A Station favorable to the pursuits of science* (2000), in particular since its reorganisation in 1817 by a new director. The most influential mathematics professor there became Charles Davies (1798–1876), who also turned out to be the American mathematics textbook entrepreneur, somewhat comparable to Lacroix – and this mainly by transmitting French textbooks.[3] Among his long list of textbooks, the first he chose to translate from the French were Legendre's geometry and Bourdon's algebra. One might ask why Davies chose to publish a translation since there already existed a translation of Legendre in the United States: by John Farrar (1779–1853), published in 1819. Actually, Farrar's text was not complete – he had omitted various parts which he had judged to be too difficult for his students at Harvard University. Davies' translation of Legendre, first published in 1828 and re-edited an enormous number of times

[3] See the advertisement for Davies' "Course of mathematics", in the 1860 edition of his Descriptive Geometry, listing 14 books of this series (Albree et al. 2000, p. 264). It is telling how his French translations were listed there: "Davies' Bourdon's Algebra", "Davies' Legendre's Geometry and Trigonometry".

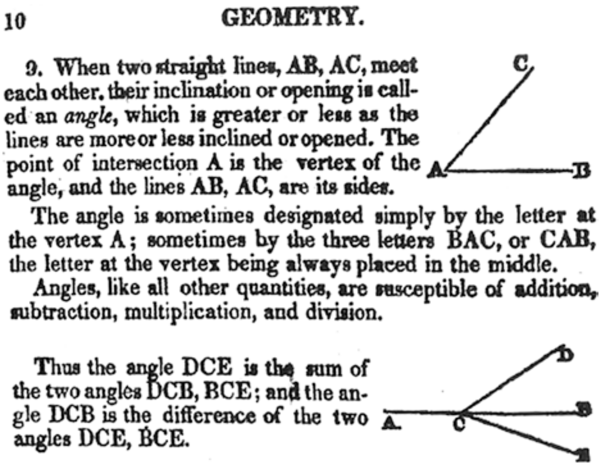
throughout the entire century, presents another pertinent case of transmission. Already in its title a specific reception and adaptation and, moreover, that it is not a direct transmission is made clear:

> Elements of Geometry and Trigonometry, translated from the French of A. M. Legendre, by David Brewster, *revised and adapted to the course of mathematical instruction in the United States* (my emphasis).

It, therefore, reveals another case of double transmission: Davies had chosen as his basis the English translation by David Brewster (1781–1868), of 1822, and revised the modifications which Brewster had applied, by proper modes of reception – in particular, regarding the use of proportion theory. So, what did Davies mean by "adapted to the course of mathematical instruction" in the USA? Surprisingly, he claimed that his translation aimed to strive for more generality than the original. This claim is due to the issue of placement of the diagrams. Davies had explained his understanding of generality in the preface of his translation:

> In the original work [..] the propositions are not enunciated in general terms, but with reference to, and by the aid of, the particular diagrams used for the demonstrations. It is believed that this departure from the method of Euclid has been generally regretted. The propositions of Geometry are general truths, and without reference to particular figures. The method of enunciating them by the aid of particular diagrams seems to have been adopted to avoid the difficulty which beginners experience in comprehending abstract propositions. But in avoiding this difficulty, and thus lessening, at first, the intellectual labour, the faculty of abstraction, which it is one of the primary objects of the study of Geometry to strengthen, remains, to a certain extent, unimproved" (Davies 1839, iii).

This criticism shows that Davies had never seen the French original, which has all diagrams placed, as already commented, on separate tables, at the end of the volume. Brewster, however, had used the other technique, namely of printing the diagrams directly accompanying the respective propositions within the text and no longer separated at the end of the work (see Fig. 10.1). Furthermore, Brewster does indeed refer directly to the diagrams next to the text – thus apparently


10 GEOMETRY.

9. When two straight lines, AB, AC, meet each other, their inclination or opening is called an *angle*, which is greater or less as the lines are more or less inclined or opened. The point of intersection A is the vertex of the angle, and the lines AB, AC, are its sides.

The angle is sometimes designated simply by the letter at the vertex A; sometimes by the three letters BAC, or CAB, the letter at the vertex being always placed in the middle.

Angles, like all other quantities, are susceptible of addition, subtraction, multiplication, and division.

Thus the angle DCE is the sum of the two angles DCB, BCE; and the angle DCB is the difference of the two angles DCE, BCE.

Fig. 10.1 Extract from Brewster's 1822 translation of Legendre's geometry, p. 10

justifying Davies' criticism and in some way provoking a misunderstanding – but it is clear that he did not want to associate the particular with the general. Indeed, what constitutes a "general figure" in geometric diagrams has continually been a subject of philosophical and didactic debates – without having had a satisfactory solution.

By the late nineteenth century, in addition to the permanent modifications introduced by Davies, due to his successor and new editor of this geometry textbook, J. Howard Van Amringe, professor at Teachers College in New York, the Legendre-Carlyle-Brewster-Davies transmission reached over 500 pages. The text then practically only minimally corresponded to the original French text of 1794.[4]

10.6 Islamic Countries

It is most remarkable that almost all reform and modernisation efforts in the **Ottoman Empire** drew on transmission from France: textbooks, teachers and sending students to France for transmitting knowledge after their return. Given that the earlier-on always victorious Ottoman Empire suffered during the eighteenth century but defeats, military engineering schools were established in Istanbul – the first in the 1730s, with French officers for training army cadets. One of its follow-up schools, at a broader scale, was the Imperial Engineering School, *Mühendishâne -i Hūmayūn*, founded in 1784, with French engineers as teachers for the military sciences and Turkish teachers, mostly graduates of the former engineering schools, for teaching mathematics and using French textbooks. The basic mathematics textbooks used were those by Bézout, Charles Bossut, and the logarithmic tables by Jean-François Callet (Abdeljaouad 2012, pp. 487 ff.). After the 1807 revolt against modernisation, Sultan Mahmud II intensified the reform policies; Bézout's textbooks are reported as continually used (ibid., p. 489).

Egypt, officially a suzerain part of the Ottoman Empire, but practically independently governed by Muhammad Ali since 1805, also practiced intense modernisation efforts in its army and education system. While at first Italy had been addressed for modernisation, by hiring Italian teachers and sending students to Italy, from 1826 it was to France that the government sent a great number of students, among other faculties, to *École polytechnique*. And it was French textbooks which were translated by those who returned from their studies. Among these translations were: the *arithmétique* of Reynaud, the *algèbre* of Mathias Mayer d'Almbert, the *géométrie* of Legendre, the *trigonométrie* of Lefébure, the *calcul différentiel et intégral* of Boucharlat and the *géométrie descriptive* of Émile Duchesne (ibid. pp. 494 f.; Crozet 2019, pp. 361 ff.).

Tunisia, likewise a suzerain part of the Ottoman Empire, felt threatened by the French occupation of Algeria in 1830 and, therefore, modernised its army and parts

[4] A series of studies about French textbook transmissions to the States have been undertaken by Thomas Preveraud (see Preveraud 2015).

of its education system. At the military schools established now, French language was taught since most of the used textbooks were written in French (Abdeljaouad 2014).

Printed geometry textbooks in Persian appeared for the first time in **Iran** in the context of the Dār al-Funūn, founded in 1851 as a type of polytechnic school, for modernising the training of officers and engineers. A modern printing press had been founded in Iran in Tabriz, around 1816. Printing mathematical books, in particular geometry ones, afforded, however, a technique allowing the printing of diagrams – the problem discussed in Sect. 3.3. The solution found in Iran was lithographic printing:

> But the use of print technology did not become widespread until the proliferation of lithographic presses about the middle of the nineteenth century. Not only was lithography a cheaper process than typography (especially if the publication involved any illustrations), the technology also allowed printers to retain some of the aesthetic characteristics of manuscript copies, which is often cited as a reason why lithography rapidly surpassed typography for printing books in Arabic and Persian, especially in Iran and India (de Young 2017, p. 90).

The earliest West-European geometry textbook translated into Persian and printed in this lithographic manner is – once again! – Legendre's geometry. The translation was published in 1856 (H 1273), but it was made of an original version – hence, not of a Blanchet-modified one (ibid., p. 91).

10.7 Germany

In Chap. 8, the profound difference between Prussia, with its neo-humanist reforms from 1810, and the other German States had been evoked. This difference remained characteristic, especially in the first two thirds of the nineteenth century, but began to attenuate from the last third of this century and even flattened out during the twentieth century, at least in many aspects. One element of this process was the fact that the exceptional role of teachers in Prussian secondary schools of also being scholars was "normalised", since the founding of the German Empire in the year 1871. Since then, the educational systems of the still numerous German States had to achieve a certain degree of homogeneity. One of the consequences was that the teachers of the *Gymnasien* took on the normal role of teaching staff; the Prussian teachers thus lost their character as scientists.

However, as the Empire had the status of a confederation, the traditional character of decentralisation of cultural and educational structures was also maintained to a high degree. An assessment of the number of mathematics schoolbooks used in secondary schools indicated, only for Prussia:

- 130 books in the year 1880;
- 132 books in the year 1890.

While these numbers show rather a continuity with the tables in Chap. 8, a certain reduction can be observed since the end of the nineteenth century:

– 100 books in the year 1899, and
– 90 books in the year 1906 (Greve and Rau 1959, p. 313).

It is probably due to this still enormous number of mathematics schoolbooks published in Germany that there did not emerge a small number of leading authors, like in France, which might have attained a certain impact in other countries. If there has been an impact of German books in some countries, it seems that this did not occur in an analogously dominant manner as with books by French authors, but more in a shared manner, shared with transmissions from other metropoles – and, hence, with France.

10.8 Russia

During the eighteenth century, Dutch, British, German, and French were used. Culture was traditionally influenced by French and German cultures. Since the restructuring of the Russian educational system from 1802, there is evidence that French mathematics textbooks have been translated and used (Karp 2014, p. 314). But there is also evidence of the influence of German books (ibid.).

10.9 Greece

When Greece became independent, there had been various connections by Greeks with mathematics in France but also in Austria (Kastanis and Kastanis 2006, pp. 522 ff.). Quite important would be that during the Napoleonic period the Ionian Islands, and among them Corfu, which should only later become a part of Greece, was governed by France, and that the French engineer Charles Dupin (1783–1873) created there the Ionian Academy, which was later directed by Ioannis Carandinos (1784–1835), who strongly influenced the politics of the emerging Greek State. Strong French influence was due to Carandinos: “he established post-revolutionary French mathematics in Greece by making it the hub of the curriculum” (ibid., p. 228). Carandino developed an intense activity for establishing Greek mathematics teaching based on French conceptions and textbooks:

> Carandinos’s syllabus at the Ionian Academy included algebra, geometry, trigonometry, descriptive geometry, infinitesimal calculus and mechanics. It is likely that he also taught arithmetic in preparatory classes in the beginning. For the purposes of his courses, he first translated parts of the Lacroix series, of Bourdon’s *Elémens d’algèbre*, Biot’s *Essai de géometrie analytique, appliquée aux courbes et aux surfaces du second ordre*, Lagrange’s *Théorie des fonctions analytiques* and Poisson’s *Traité de mécanique*. Besides, he used Monge’s *Géométrie descriptive*, and Lacroix’s *Traité élémentaire de trigonométrie rectiligne et sphérique, et d’application de l’algèbre à la géométrie*. Later, he translated and

> published Bourdon's *Elémens d'arithmétique* (Vienna, 1828), Legendre's *Elémens de géométrie* (Corfu, 1829), John Leslie's *Geometrical Analysis* (Corfu, 1829), and Legendre's *Treatise on Trigonometry* (Corfu, 1830). (ibid., pp. 528 f.).

Besides the Austrian influence, dating from the eighteenth century, there were, however, also transmissions from Germany. Their political context was that Greece, after having gained independence from the Ottoman Empire, was under control of the European powers. And then it was decreed that Greece should be a monarchy, and eventually it was decided that it should be governed by one of the minor dynasties in Europe – for not creating additional problems. The imposed dynasty was the Bavarian one: the young Otto von Wittelsbach (1815–1867) became, in 1832, King of Greece. This evidently implied, by now, German influences on educational policy, so that a first period of the development of mathematics teaching in Greece, up to about 1880, "can be characterized as one of foreign influence, since translations of textbooks very current at the time in France and Germany prevailed" (Zormbala 2002, p. 201). The German influence is mainly due to Georgios Gerakis who had received in 1837 a scholarship by the ministry to study in Germany. Upon his return, he translated three schoolbooks, on arithmetic and algebra, on plane geometry and on stereometry by Karl Koppe (1803–1874), mathematics teacher at the Gymnasium in Soest (Prussia).

The translations by Gerakis were widely distributed in Greece and were used in schools for a long time: his Arithmetic was officially prescribed as a school book by the Greek Ministry of Education, from 1863 to 1877; Geometry, first published in 1855 and appearing in five editions by 1877, was the dominant textbook in the Hellenic schools – roughly the lower grades of secondary schools. It has not yet been possible to determine how Gerakis came to know of Koppe's work, being in Prussia of rather regional importance for the province of Westphalia (Schubring 2010, pp. 148 ff.), and effected this remarkable transmission to Greece (see Kastanis and Kastanis 2006, p. 532).

Given the decentralised and diversified political structure of Germany and the multiplicity of cultural centres, this transmission from Germany to Greece presents a revealing case of how a textbook of regional importance at the origin can achieve a strong impact upon the reception in another structure.

10.10 The Netherlands

In The Netherlands, there had been a long-standing tradition of publishing mathematics textbooks, mainly following the practice-oriented conceptions of teaching (see Sect. 3.2.). During the period of the Batavian Republic (1795–1806) and the following periods of a Napoleonic satellite State (1806–1810) and the incorporation into the Napoleonic Empire (1810–1814), there had been more contact with French textbooks. From the establishment of the Kingdom of the Netherlands in 1815, however, there had not been much use of French textbooks – Lacroix's geometry is mentioned for the Polytechnic in Delft (Smid forthcoming). German textbooks were also of influence but more rarely (ibid.). Since the Dutch government basically refrained from interfering with educational matters, there were in general no schoolbooks prescribed. Textbook production was hence largely in the hands of Dutch mathematics teachers. An assessment of the books in use at the lower grades of the HBS – *Hogere Burgherschol*, a genuinely Dutch type of *Realschule* – reveals the telling practice that mathematics teachers, as in Germany, preferred proper, local schoolbooks:

> In 1890 a mathematics teacher from the HBS in Alkmaar, W.F. Koppeschaar jr., published an overview of the books used in the three lower grades of the HBS […]. One of his results was that for arithmetic 12 different books were in use, for algebra 18 and for plane geometry 12. Many titles were used at only one or two schools, some titles on some more, but there was one textbook author whose books were by far the most in use (ibid. p. [143]).

And Smid suggests that this is due to "teachers who wanted to use a book tailored to their own preferences" (ibid., p. [123]). The most successful textbook author was the influential mathematics teacher and educator Jan Versluys (1845–1920), see (ibid., p. [144] ff.).

10.11 England

While the Dutch textbook production occurred basically in an independent manner but sharing the conceptions of school mathematics on the "Continent", it is well-known that England practiced a very special conception: as said by Augustus de Morgan (1806–1871), Euclid is "a very English subject". Euclid's *Elements* determined, by various English translations and adaptations, the academic system in England, being the subject of the final examinations. Attempts in the first half of the nineteenth century to open England to the "Continent", in particular by the Cambridge-based *Analytic Society*, had no lasting effect.

In the second half of the century, an association was created for reforming Euclid:

> The emphasis on Euclid, and a growing feeling that it was outdated and inappropriate, led in 1871 to the creation of the Association for the Improvement of Geometrical Teaching (AIGT), probably the world's first subject teachers association. As its name suggests, the AIGT argued for a replacement for Euclid, an aim already dismissed by a committee of the British Association set up in 1869 (including Cayley, Clifford and Sylvester) which thought nothing so far produced 'is fit to succeed Euclid'. The AIGT produced its own course, but it was not to prove a success or be widely welcomed by universities (Howson 2014, p. 261).

The AIGT changed its name in 1897 to Mathematical Association (MA) and set out for new efforts, which eventually achieved certain reforms in 1902. Then, pressure for change and ensuing movements were evidenced by numerous initiatives, committees, and reports. Admittedly, the reforms were at first quite conservative – changes in the curricula and examination procedures were understood as effecting "various omissions and simplifications" (Price 1994, p. 58). Changes meant to abandon the Euclidean approach as the only one, as shown by formulations like "any solution which shows an accurate method of geometrical reasoning will be accepted" (ibid., p. 60). Cambridge, where major resistance had prevailed, determined a "tolerance" line for elementary mathematics teaching:

(1) In demonstrative geometry, Euclid's Elements shall be optional as a text-book, and the sequence of Euclid will not be enforced. The examiners will accept any proof of a proposition which they are satisfied forms part of a systematic treatment of the subject.
(2) Practical geometry is to be introduced, along with deductive geometry, and questions will be set requiring careful draughtsmanship and he use of efficient drawing instruments.
(3) In arithmetic, the use of algebraic symbols and processes will be permitted.
(4) In algebra, graphs and squared paper will be introduced [...] (ibid., pp. 60 f.).

In the following years, textbooks implementing these reform lines became popular, in particular that by Charles Godfrey (1873–1924), one of the main reform activists in the MA, and Arthur Warry Siddons. Their *Modern Geometry* (1908), judged as a "rethought and pupil-centred, 'watered-down' Euclid" (Howson 2014, p. 262), continued to be used in some schools, even for the next 60 years.

Chapter 11
An Outlook to the Twentieth Century

In this concluding chapter, a few systematic issues should be highlighted.

11.1 Segmentation and 'Nationalisation' of Mathematics to Be Taught

The nineteenth century turned out to be decisive, due to the institutionalisation of public education systems, practically not only for the process of attenuating national differences but especially for the increasingly rigid separation between the different levels of education taking place in European countries. In particular, the separation and delimitation between secondary schools and higher education – universities and higher technical training institutions – were definitively established, even in countries where a genuine secondary education had not existed before, such as Spain and England. A consequence of this separation was not only that transitions of teachers from schools to professor positions at universities occurred ever more rarely, but also that textbooks were now clearly divided into different types for the various levels of education. There were no longer general textbooks that could serve equally well at one level or another. Thus, the type of textbook for secondary schools and primary schools was definitively established, distinguishing itself from textbooks for higher education.

During this process of the nineteenth century, where the current structures of the educational system emerged, with the particular characteristics of each level, the production of textbooks became even more adapted to the requirements of these levels: it can be said, therefore, that the production was "nationalised"; the great classical textbooks serving in a "supra-national" dimension, by transmission, translation, and reception, as standards in different countries – Euclid, Bézout, Euler, Legendre, Lacroix – began to disappear, in general since the beginning of the

G. Schubring, *Analysing Historical Mathematics Textbooks*, International Studies in the History of Mathematics and its Teaching,
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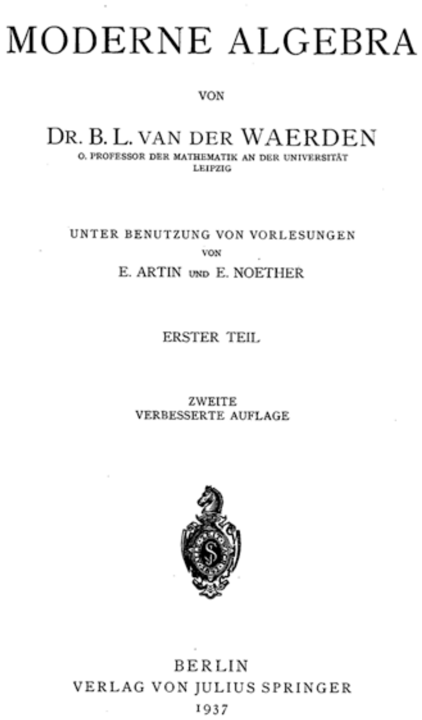
twentieth century. But it is important to underline the immense importance of these standard textbooks that effected the transmission of mathematics to countries where there had not yet existed an active community of mathematicians and which thus contributed to the creation of a mathematical culture.

11.2 Interfaces Between School Mathematics and Academic Mathematics

Furthermore, also the productive role of textbooks in instigating new developments in mathematical science continued: indeed, this role revealed hitherto unknown dimensions of impact. It will suffice to mention two examples.

The first example affects one subdiscipline of mathematics: algebra. Bartel L. van der Waerden (1903–1996) transformed the completely novel concepts of algebra, communicated by his teacher Emmy Noether (1882–1935) in her lectures at Göttingen university, into the textbook *Moderne Algebra*, first published in 1930/31 (Fig. 11.1). It effectively modernised algebra, by generalising it at a higher abstract level, and changed the research approach (and teaching) in algebra globally, instigating many innovations.


MODERNE ALGEBRA

VON

DR. B. L. VAN DER WAERDEN
O. PROFESSOR DER MATHEMATIK AN DER UNIVERSITÄT
LEIPZIG

UNTER BENUTZUNG VON VORLESUNGEN
VON
E. ARTIN UND E. NOETHER

ERSTER TEIL

ZWEITE
VERBESSERTE AUFLAGE

BERLIN
VERLAG VON JULIUS SPRINGER
1937

Fig. 11.1 Cover of the second edition of *Moderne Algebra*, of 1937

The second example concerns not just one subdiscipline but the whole of mathematics – and affected thereafter mathematics education globally: it is the work of the collective Nicolas Bourbaki. The collective, which chose as its name the pseudonym Nicolas Bourbaki – and which is always rejuvenated by new generations of research mathematicians – was founded in 1934 at a meeting in a café in the Quartier Latin in Paris by graduates of the *École Normale Supérieure*, who were dissatisfied with the analysis textbook by Édouard Goursat (1858–1936) used at the institution and regarded it as conceptually obsolete. This collective elaborated a textbook-series with the already ambitious title: *Éléments de Mathématique*.

Once beginning the work, the group widened the scope to restructure the entire architecture of mathematics. In search of fundamentals for restructuring Bourbaki determined set theory, the novel foundational discipline developed by Georg Cantor since the 1870s as their basis. The first volume should, therefore, appear to be the volume on set theory, *Théorie des ensembles*. It turned out to be a difficult challenge. In 1939, the first publication of Bourbaki was on set theory, but this *Fascicule de résultats* was just an abridged version of the projected book (Fig. 11.2). The volume itself was published only in 1970. The Second World War interrupted the work considerably, but after the war, the phase of main and international work and impact began. In 1948 the programmatic chapter *L'Architecture des Mathématiques* (Bourbaki 1939) disseminated this conception.

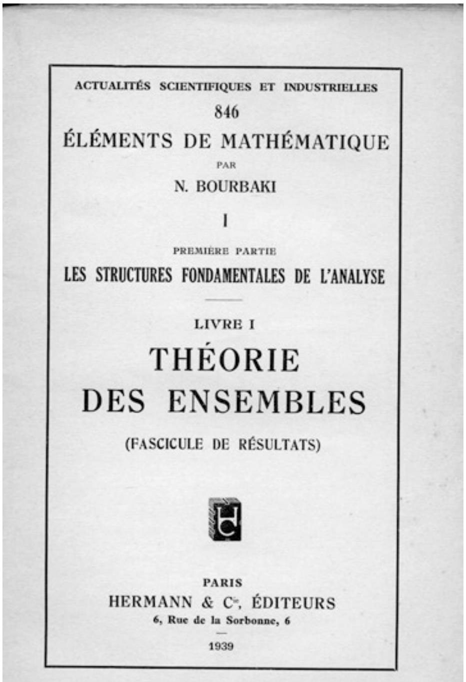

ACTUALITÉS SCIENTIFIQUES ET INDUSTRIELLES
846
ÉLÉMENTS DE MATHÉMATIQUE
PAR
N. BOURBAKI
I
PREMIÈRE PARTIE
LES STRUCTURES FONDAMENTALES DE L'ANALYSE
LIVRE I
THÉORIE
DES ENSEMBLES
(FASCICULE DE RÉSULTATS)
PARIS
HERMANN & Cie, ÉDITEURS
6, Rue de la Sorbonne, 6
1939

Fig. 11.2 Cover of the first publication of the Bourbaki group; the results of set theory, of 1939

As is well known, the project developed in a way that served not only to elaborate textbooks for French higher education but to modernise mathematics with a global influence on research and even on teaching at all its levels. It was the mathematics curricular reform movement, known as *Modern Math* or as *New Math*. The movement was launched in 1959 by the Royaumont Seminar as apparently being international, but in this respect restricted to the "West" as a by-product of the Cold War, instigated by the Sputnik shock of 1957. However, not only had the movement already been prepared by various regional activities for modernising the mathematics curricula since the 1950s but also the "East" experienced quite analogous reform activities.

Regarding the widespread claim of Modern Math as having globally homogenised the teaching of mathematics, Jeremy Kilpatrick has stated, in his evaluation of the international effects: "The more school mathematics is internationalized, the more clearly its national character is revealed" (Kilpatrick 2012, p. 570).

Yet, one has to state a lasting and rather general effect of the reform movement. The strict social, conceptual, and epistemological separation into three different corpora of primary reckoning and geometrical rudiments, of secondary school mathematics and of academic mathematics has been substituted by the recognition that there is but one mathematics!

11.3 Who Are the Textbook Authors?

In textbook analysis, the issue of who are the authors of textbooks is rarely addressed. This issue turns out to become relevant essentially since the establishing of public-school systems, resulting in the trifold structure of primary, secondary, and higher education. And it is international comparison, which calls attention for differing patterns regarding textbook authorship.

While there used to be a broad but unstructured sample of categories of authors, ranging from practitioners, amateurs, clerics, encyclopedists, to teachers, and scientists, this sample became basically restricted, in the wake of the French Revolution, to just two categories: teachers and scientists as higher education professors. And as was occasionally noted for textbook production regarding some countries in the nineteenth century, one remarks only school teachers as authors, while one finds in other countries just university professors or academicians as authors. This structural difference seems to be due to a decisive point in the respective educational system: which educational subsystem is provided with the authority to allow students the transition from secondary to higher education. In educational systems where it is the secondary school which entitles students to access higher education – as by the German "Abitur" – schoolbook production teams will be composed essentially by teachers. In educational systems, where, however, it is higher education institutions which defer the right of access – like the "bac" in France – authors are used to being mathematics scientists. In Russia, where it is traditionally academicians who write schoolbooks, and for where there was until recently a separate exam for higher education access, A. N. Kolmogorov, for instance, edited schoolbooks: *Geometry* (1979) for grades 6–8, and *Algebra and Beginning Analysis* (1976) for grades 8–9.

11.4 The Growing Diversification of Textbook Material

In a final survey, one cannot fail to highlight the growing diversification of the media type textbook, in particular to comment upon its drastic differentiation since the beginning of the digital age. Traditionally, there was just one unique type of material – from the manuscript, which was only in the hands of the teacher, to the printed book, which might be used by the teacher and the student. We noted the innovation, proposed by Condorcet in 1792, to have a separate textbook for the teacher – as a guide in the method of teaching. And we noted a differentiation, which had occurred in the nineteenth century, with a triple medial form: a main textbook, a guide for the teacher, and a collection of exercises for the student. This structure was basically maintained until the 1960s and the 1970s, when the methodisation of schoolbooks, beginning with printing editions with coloured illustrations, intended to make learning more joyful. But a multiple structure of the main textbook had actually been introduced already earlier on. As it became evident with Lacroix's "Cours de Mathématiques" and his followers, the textbook could consist of various volumes for the different subdisciplines of school mathematics, or in a parallel manner for the ascending grades of the school structure.

Differing media forms for school learning had been introduced at first by British Modern Math projects: the working cards, for instance of the Nuffield Mathematics Project, were intended for individualised handling of learning tasks by students, using the laboratory technique of "activity cards". Later, more media forms for individualised handling were developed, accompanying the progress of computing devices, with schoolbook series becoming a set of printed material and of material on CD-Rom.

The difference will be shown here with the editions of a textbook series, popular in Western Germany since the 1950s and still on the market in the 2010s, in profoundly revised versions. The series is known as the "Lambacher-Schweizer", written by Theophil Lambacher and Wilhelm Schweizer, and published by Klett, since the late 1940s in the Federal Republic of Germany. Its original title was *Mathematisches Unterrichtswerk* (mathematical schoolbook series), with the addition: "für höhere Schulen" – for secondary schools. At present, the title of the series is *Mathematik für Gymnasien*.

In the 1950s, the series was structured according to the three traditional levels of German *Gymnasien*: *Unterstufe* (grades 5–7), *Mittelstufe* (grades 8–10), *Oberstufe* (grades 11–13). The titles of each part of the series indicated the respective branch of school mathematics:

- 1.1. *Rechnen und Raumlehre*[1] *1*
- 1.2. *Rechnen und Raumlehre 2*
- 2.1.1. *Algebra 1*
- 2.1.2. *Algebra 2*
- 2.2.1. *Geometrie 1*
- 2.2.2. *Geometrie 2*
- 3.1. *Analysis*
- 3.2. *Analytische Geometrie*
- 3.3. *Kugelgeometrie*

Thus, for the 9 years of Gymnasium, the series provided nine books. The situation in the 2010s proves, however, to be drastically different. First of all, Klett published the series now in 17 different editions, according to the differing curricula of the Federal States of the Federal Republic. I will now show the editions for the State Nordrhein-Westfalen, since Bielefeld belongs to it. Former *Unterstufe* and *Mittelstufe* were meanwhile shortened to 5 years, grades 5–9. And the materials sold by Klett for grade 5 are:

- one printed schoolbook for the student
- two versions of an ebook, consisting of, besides the book, digital form of a set of additional media:
- Multimedial Enichments, containing:
  - 15 didactically edited Lambacher-Schweizer explanatory films
  - 2 adaptive educational films with interactive exercises and individual assistance
  - 50 interactive algebra-modules, with step-by-step interactive solution approaches
  - 84 interactive task sequences for practicing
  - 12 interactive test sequences
- three *workbooklets* (*Arbeitshefte*): working material for the student, with solutions; one of them with additional learning software, and a *Klassenarbeitstrainer,* training module for in-class-tests
- one teacher guide: *Handreichungen für den Unterricht*.

The same number of media is destined for grade six, grade seven, grade eight, and grade nine.

For the *Oberstufe*, now grades 10–12, there are thematic workbooks:

- Arbeitsbuch Stochastik,
- Arbeitsbuch Analysis I,
- Arbeitsbuch Analysis II,

[1] "Rechnen" meant an elementary form of arithmetic, and "Raumlehre" meant an elementary form of geometry.

– Arbeitsbuch Analytische Geometrie, workbooklets
– each one with “Erklärfilme”, explaining videos, and
– two Abi-Coaches: training material for the final exam Abitur.
– This enormously increased number of units and the new variety of media evidence a kind of media revolution in the production of schoolbooks.

11.5 The Textbook Triangle Reassessed

Despite structural changes over the course of the historical development exposed here, there is one characteristic that persists: it is, on the one hand, the intimate connection between the teacher and the textbook and, on the other hand, the opposition or dichotomy between oral and written teaching. Undoubtedly, this dichotomy is no longer specific to certain states or nations; and it also no longer affects the level of secondary schools. As practically all studies on the teaching of mathematics evidence, the reality of this teaching shows that textbooks are the main recourse for teachers. The two opposite poles did manifest themselves in the different forms of practicing the textbook triangle – the contrast between its practice in nineteenth century France and in nineteenth century Prussia.

Thus, the traditions of orality in teaching that were characteristic for classrooms were manifest in Prussian Gymnasien and universities in the nineteenth century. The exemplary case there was presented by Weierstraß: he did not publish his new approaches and methods for research on analysis and function theory. Those who wished to learn this new knowledge had to attend his lectures. Often, Weierstraß criticised people who published expositions according to lectures they had attended because they distorted his concepts. A revealing alternative is found in Ulisse Dini’s famous textbook on the foundations of the theory of functions (1878/1892). Dini quoted Weierstraß’s methods and results several times, being able to give as reference only: “Satz von Weierstrass (in dessen Vorlesungen)” – proposition by Weierstraß (in his lectures) (Dini 1892, p. 29). On another occasion, Dini expressed: “...(es) wird zweckmäßig sein, hier den folgenden Satz von Weierstrasys wiederzugeben”, with the footnote “Aus dessen Vorlesungen” – (it) will be useful to reproduce here the following theorem by Weierstrass, taken from his lectures (Dini 1892, p. 57).

Another means of referencing orality was used in the likewise important textbook by Stolz and Gmeiner on theoretical arithmetic: for crediting Weierstraß’s notions of negative and complex numbers, they referred to two French books; their authors had been the first to expose these concepts, having attended Weierstraß’s lectures (Stolz and Gmeiner 1911, p. 78).

How often has one not heard to claim that continuing the university pattern of lectures for the principal disciplines of mathematics should be superfluous and outdated, given that excellent textbooks are accessible so that students should learn from them! Actually, in the period of expanding the universities and constructing new ones in the 1960s and 1970s, there were even some universities built with almost no lecture halls and rather a lot of seminar rooms. After a few years, these ‘innovations’ proved to be counter-productive. Lectures could not be replaced by the degenerate textbook-triangle, with just the two poles of textbook and teacher.

These experiences confirmed once more that orality and textbook are inseparable. The triangle has to be maintained with its three vertices.

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